Tutte trails of graphs on surfaces

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Abstract

A *Tutte trail* T of a graph G is a trail such that every component of $G \backslash V(T)$ has at most three edges connecting it to T. In 1992, Bill Jackson conjectured that every 2-edge-connected graph G has a Tutte closed trail.

In this thesis, we show that Jackson's conjecture is true when G is embedded on the plane and the projective plane. We also give some partial results when G is embedded on the torus.

Acknowledgements

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Last but not the least, I would like to thank my family: my mother and sister for supporting me spiritually throughout writing this thesis and my my life in general.

In memory of my father, Supon Sinna.

Contents

Chapter 1

Introduction

In this thesis, we consider finite undirected graphs which may have multiple edges but no loops. Finding a Hamilton cycle is one of the most popular subjects in graph theory. Two of the significant results on Hamilton cycles concern planar graphs: Whitney [37] proved that every 4-connected planar triangulation has a Hamilton cycle, and Tutte [36] extended this to all 4-connected planar graphs. Tutte [35] showed that the *Tutte graph* (see Fig. 1.1) is a counterexample to Tait's Conjecture that every 3-connected cubic plane graph has a Hamilton cycle.

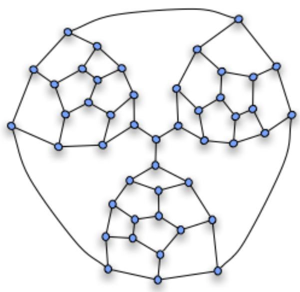


Figure 1.1: Tutte graph.

A *vertex-Tutte cycle (path)* P in a graph G is a cycle (path) such that either P is a Hamilton cycle of G, or every component of $G \backslash V(P)$ has at most three neighbors on P. Tutte [36] showed that every 2-connected plane graph has a vertex-Tutte cycle, and deduced that every 4-connected plane graph has a Hamilton cycle. Extending Tutte's technique, Thomassen [32] and Sanders [29] showed that every 2-connected graph G has a vertex-Tutte path from u to v containing e for each $u, v \in V(G)$ and $e \in E(G)$, and deduced that every 4-connected planar graph is Hamilton-connected, i.e., there is a Hamilton path connecting any two vertices.

The complete bipartite graph $K_{m,n}$, where $4 \leq m < n$, is an example of a non-Hamiltonian 4-connected non-regular graph. Meredith [23] showed that the *Meredith graph* (see Fig. 1.2) is a counterexample to Nash-Williams' Conjecture that every 4-connected 4-regular graph has a Hamilton cycle.

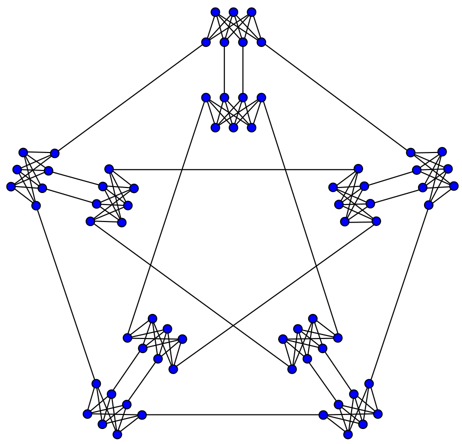


Figure 1.2: Meredith graph.

A *dominating trail (cycle)* of a graph G is a trail (cycle) of G containing at least one endpoint of every edge of G. A graph G is *essentially k-edge-connected* if G is connected and does not have an edge cut X of size less than k for which all components of $G \backslash X$ have at least one edge.

There are several conjectures about Hamilton cycles in 4-connected graphs and dominating closed trails in essentially 4-edge-connected graphs by various mathematician which now are known to be equivalent.

Theorem 1.1 *All of following statements are equivalent:*
(A1) Every 4-connected claw-free graph is Hamiltonian (Matthews and Sumner [22]);
(A2) Every 4-connected line graph is Hamiltonian(Thomassen [33]);
(A3) Every essentially 4-edge-connected graph has a dominating closed trail;
(A4) Every essentially 4-edge-connected cubic graph has a dominating cycle(Ash and Jackson [1]);

(A5) Every essentially 4-edge-connected cubic graph that is not 3-edge-colorable has a dominating cycle (Fleischner [10]).

Matthews and Sumner [22] conjectured (A1). Thomassen [33]) independently conjectured (A2). Since every line graph does not contain a claw ($K_{1,3}$), (A1) implies (A2). Ryjáček [26] defined that a graph H is a closure of a claw-free graph G denoted $H = cl(G)$, if:
(i) there is a sequence of graphs $G_1, ..., G_t$ such that $G_1 = G, G_t = H, V(G_{i+1}) = V(G_i)$ and $E(G_{i+1}) = E(G_i) \cup \{uv : u, v$ are neighbors of x_i and $uv \notin E(G_i)\}$ for some $x_i \in V(G_i)$ such that $\{y \in V(G_i) : x_i y \in E(G_i)\}$ induces a connected noncomplete graph for $i = 1, 2, ..., t-1$;
(ii) For each $x \in V(H)$, if $N_H(x) = \{y \in V(H) : xy \in V(H)\}$ induces a connected graph, then $N_H(x)$ is a complete graph.
Ryjáček showed that for all claw-free graphs G, G is Hamiltonian if and only if $cl(G)$ is Hamiltonian. By using the closure of G, Ryjáček showed that (A2) implies (A1).

Harary and Nash-Williams [9] showed that the line graph of G has a Hamilton cycle if and only if G has dominating closed trail. Since it can be checked that $L(G)$ is k-connected if and only if G is essentially k-connected, (A2) and (A3) are equivalent.

Ash and Jackson [1] conjectured (A4). Fleischner and Jackson [11] proved that (A4) is equivalent to (A2).

Fleischner [10] conjectured (A5). Kochol [18] proved that (A5) and (A4) are equivalent.

A graph G is *1-Hamilton-connected* if $G - x$ is Hamilton-connected for every $x \in V(G)$. There are also some equivalent conjectures about Hamilton-connected graphs.

Theorem 1.2 *All of following statements are equivalent:*
(A2) Every 4-connected line graph is Hamiltonian;
(A6) Every 4-connected line graph is Hamilton-connected (Kužel and Xiong [19]);
(A7) Every 4-connected claw-free graph is Hamilton-connected (Ryjáček and Vrána [27]);
(A8) Every 4-connected line graph is 1-Hamilton-connected(Kužel, Ryjáček, and Vrána [20]).

Kužel and Xiong [19] conjectured (A6) and showed that (A6) is equivalent to (A2). Ryjáček and Vrána [27] conjectured (A7) and showed that (A6) and (A7) are equivalent. Kužel, Ryjáček, and Vrána [20] conjectured (A8) and showed that (A8) is equivalent to (A6).

For a vertex-Tutte cycle in 2-connected graphs, Jackson [13] conjectured the following statement.

Conjecture 1.3 *(A9) Every 2-connected claw-free graph G has a vertex-Tutte cycle*

Čada, Shiba, Ozeki, Vrána, and Yoshimoto [5] showed that (A1) and (A9) are in fact equivalent, and showed the following Conjecture is equivalent to (A9).

Conjecture 1.4 *(A10) Every 2-connected claw-free graph G has a vertex-Tutte cycle C such that $C = C'$ or $C \nsubseteq C'$ for every cycle C' in G.*

They also gave an example of a 3-connected claw-free graph G such that no longest cycle of G is a Tutte cycle of G.

A *Tutte trail* T in a graph G is a trail such that either T is a spanning trail of G, or every component of $G\backslash V(T)$ has at most three edges connecting it to T. Jackson [13] conjectured the following statement.

Conjecture 1.5 *(A11) Every 2-edge-connected graph G has a Tutte closed trail.*

Since a Tutte closed trail is a dominating closed trail in an essentially 4-edge-connected graph, (A11) implies (A3). Čada, Shiba, Ozeki, Vrána, and Yoshimoto [5] showed the following Conjecture is equivalent to (A3).

Conjecture 1.6 *(A12) Every essentially 2-edge-connected graph G has a Tutte closed trail.*

Since a k-edge-connected graph is essentially k-edge-connected, (A12) implies (A11), and both are equivalent to (A3).

Čada, Chiba, Ozeki, Vrána, and Yoshimoto [6] showed that two weaker versions of Thomassen's Conjecture are also equivalent.

Theorem 1.7 *The following statements are equivalent:*
(B1) Every 4-connected line graph which minimum degree at least 5 is Hamiltonian;

(B2) There exists a constant c_0 with $0 < c_0 \leq 1$ such that every essentially 4-edge-connected cubic graph G has a cycle of length at least $c_0|V(G)|$. (Bondy's Conjecture. See [11].)

Catlin [7] and Jaeger [14] showed that every 4-edge-connected graph has a spanning closed trail. Extending Catlin and Jaeger's technique, Xu, Chen, Lai and Zhang [38] showed that every 4-edge-connected graph G has a spanning trail from e_1 to e_5 containing e_2, e_3, and e_4 where $e_i, 1 \leq i \leq 5$, are all distinct edges of G. Paulraja [25] showed that a connected graph has a spanning closed trail when every edge belongs to a cycle of length 3, and also showed that a connected claw-free graph has a spanning closed trail when every edge belongs to a cycle of length at most 5. Lai [21] showed that a connected graph has a dominating closed trail when every edge belongs to a cycle of length at most 4. Holub and Xiong [12] conjectured that a 2-connected graph has a dominating closed trail when every edge belongs to a cycle of length at most 5.

For projective plane graphs, Thomas and Yu [31] showed that every 2-connected projective plane graph G has a vertex-Tutte cycle containing e where $e \in E(G)$, and this implies that a 4-connected projective plane graph is Hamiltonian. Extending Yu's result, Kawarabayashi and Ozeki [15] showed that every 2-connected projective plane graph has a vertex-Tutte path from x to y for any distinct vertices x, y of G. This implies that a 4-connected projective plane graph is Hamilton-connected.

For toroidal graphs, Grünbaum [8] and Nash-Williams [24] conjectured independently that every 4-connected toroidal graph is Hamiltonian. An *extended-vertex-Tutte cycle (path)* P in a graph G is a cycle (path) such that either P is a Hamilton cycle of G, or every component of $G \backslash V(P)$ has at most four neighbors on P. The closest result to Grünbaum and Nash-Williams' Conjecture was shown by Yu [30]: every 2-connected toroidal graph G has an extended-vertex-Tutte cycle containing e for any $e \in E(G)$. This implies that a 5-connected toroidal graph is Hamiltonian. Moreover, Kawarabayashi and Ozeki [16] proved that every 4-connected triangulation of the torus has a Hamilton cycle. Thomassen [32] showed that there exists a 4-connected toroidal graph H such that no Hamilton cycle contains e for some $e \in E(H)$. His example is obtained from a cartesian product $C_n \times C_n$ where n is even and $n \geq 4$ by adding an edge e between two non-adjacent vertices in some face of $C_n \times C_n$. (See Fig. 1.3.) Since $C_n \times C_n$ is a bipartite graph with independent sets X, Y and $|X| = |Y|$, then a Hamilton cycle of H must alternate between X and Y. This implies that there is no Hamilton cycle in H containing e. Moreover, H is not Hamilton-connected since there is no Hamilton path between the end vertices of e. Extending Yu's result, Kawarabayashi and Ozeki [17] improved the techniques in [15] and showed that a 5-connected toroidal graph is Hamilton-connected.

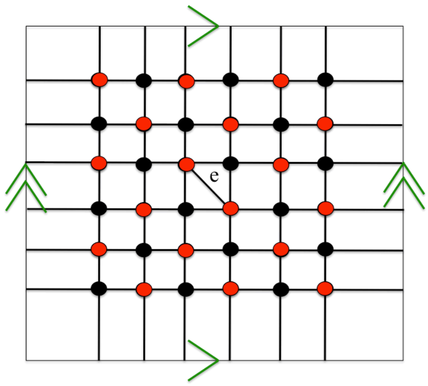


Figure 1.3: The graph $H = (C_6 \times C_6) \cup \{e\}$ embedded on torus.

Nash-Williams [24] conjectured that every 4-connected graph embedded on the Klein bottle is Hamiltonian. Brunet, Nakamoto, and Negami [4] showed that every 5-connected triangulation on the Klein bottle is Hamiltonian. Let H be the graph embedded on the Klein bottle in Figure 1.4. Then H is 4-connected, and, by the same argument as in the last paragraph, there is no Hamilton cycle containing the edge e.

For a graph embedded on an arbitrary closed surface. Thomassen [34] conjectured that every 5-connected triangulation of an orientable surface with sufficiently large representativity is Hamiltonian. Yu [39] proved this Conjecture and extended the result to any closed surface. More precisely, Yu showed that every 5-connected triangulation on a closed surface of Euler genus g has a Hamilton cycle if the triangulation has representativity at least $96(2^g - 1)$.

In this thesis, we show that Jackson's conjecture (A11) is true when G can be embedded in the plane or in the projective plane. We also give some partial results when G is embedded on the torus. The main results of this thesis are as follows.

Theorem 1.8 *Let G be 2-edge-connected graph.*
(i) If G is a plane graph with outer walk F_G, then G has an F_G-Tutte trail from u

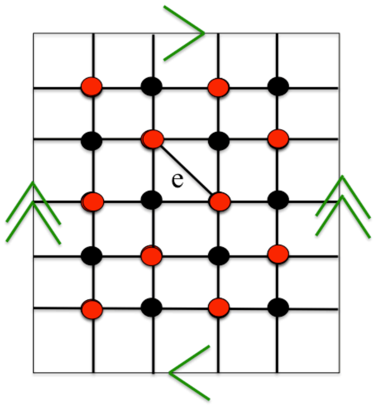


Figure 1.4: The graph H embedded on Klein bottle.

to v containing e for any $u, v \in V(G)$ and $e \in E(F_G)$.
(ii) If G is a projective plane graph, then G has a Tutte trail from u to v for any $u, v \in V(G)$.
(iii) If G is a toroidal graph with representativity at most two, then G has a Tutte closed trail.

We prove (i), (ii) and (iii) in Chapter 3, Chapter 4 and Chapter 5, respectively.

Chapter 2

Definitions and Notations

We follow Bondy and Murty [3] for terms and notations. We consider finite undirected graphs which may contain multiple edges but no loops. A *walk* in a graph is an alternating sequence of vertices and edges which begins and ends with vertices. Moreover, a *closed walk* is a walk for which its end vertices are the same. A *trail (closed trail)* is a walk (closed walk) in which all edges are distinct.

First, we define a Tutte subgraph as follows.

Definition 2.1 *Let G be a graph and F be a subgraph of G. An F-*Tutte subgraph *of G is a subgraph H of G such that*
(i) Each component of $G \backslash V(H)$ has at most three edges connecting it to H, and
(ii) Each component of $G \backslash V(H)$ containing a vertex of F has at most two edges connecting it to H.
*Moreover, H is called an F-*Tutte trail (closed trail) *if H is a trail (closed trail). Furthermore, H is called a* Tutte subgraph *if $F = \emptyset$. (See Fig. 2.1.)*

A *dominating trail* is a trail containing at least one endpoint of every edge of a graph. Note that a Tutte trail T in an essentially 4-edge-connected graph is a dominating trail since every component of $G \backslash V(T)$ is trivial.

A pair (G_1, G_2) of subgraphs of G is called a *k-seperation* of G if $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, $|V(G_1)| \geq k+1$,$|V(G_2)| \geq k+1$ and $|V(G_1) \cap V(G_2)| = k$. Note that G is *k-connected* if and only if G has no l-seperation for $l < k$ and $|V(G)| \geq k+1$.

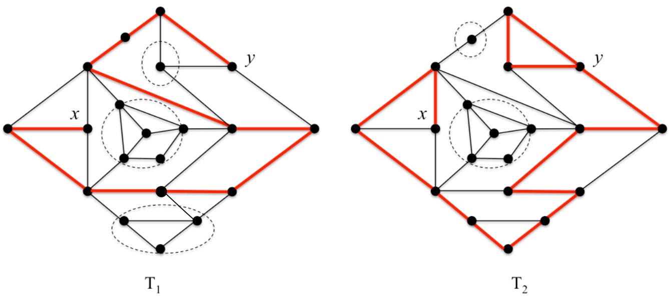


Figure 2.1: T_2 is a C-Tutte trail of a graph G when C is the outer cycle of G. T_1 is a Tutte trail of G but not a C-Tutte trail of G.

Let G be a connected graph and $S \subseteq E(G)$. S is called an *edge cut* of G if $G \backslash S$ is disconnected. A graph G is *k-edge-connected* if G is connected and does not have an edge cut of size less than k. A graph G is *essentially k-edge-connected* if G is connected and does not have an edge cut X of size less than k for which all components of $G \backslash X$ have at least one edge. (See Fig. 2.2.) Note that a k-edge-connected graph is essentially k-edge-connected.

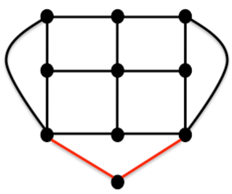


Figure 2.2: G is 2-edge-connected and essentially 4-edge-connected.

A *block* is a maximal 2-connected subgraph of G. An *edge-block* is a maximal 2-edge-connected subgraph of G. Note that an edge-block may be a single vertex. Let $B_1, B_2, ..., B_m$ be all the edges-blocks of G. We define the *edge-block-tree* graph $T(G)$ of a graph G as the graph with $V(T(G)) = \{b_1, b_2, ..., b_m\}$ and $E(T(G)) = \{b_i b_j | cd \in E(G) \text{ where } c \in V(B_i) \text{ and } d \in V(B_j), 1 \leq i, j \leq m\}$. (See Fig. 2.3.)

Next, we introduce a chain of blocks and a chain of edge-blocks as follows.

Definition 2.2 *A* chain of blocks *R is a sequence $u_0, A_1, u_1, A_2, ..., u_{n-1}, A_n, u_n$ where $A_1, A_2, ..., A_n$ are 2-connected graphs, $u_0 \in V(A_1), u_n \in V(A_n)$, $V(A_i) \cap$*

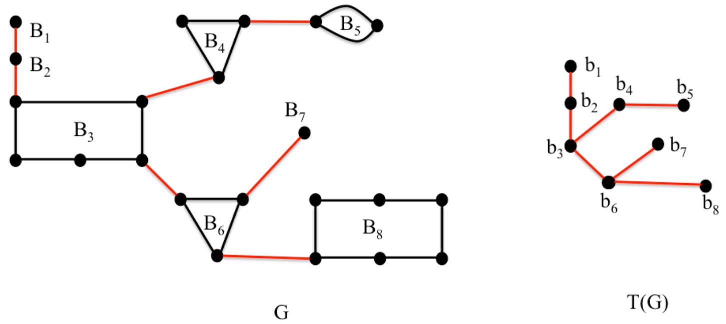


Figure 2.3: The structure of G and $T(G)$.

$V(A_{i+1}) = \{u_i\}$ *for all* $1 \leq i \leq n-1$, *and* $V(A_i) \cap V(A_j) = \emptyset$ *for all* $1 \leq i \leq j-2 \leq n-2$. *Let* $G_R = \bigcup_{i=1}^{n} A_i$.

A chain of edge-blocks S *is a sequence* $x_1, B_1, e_1, B_2, ..., e_{k-1}, B_k, y_k$ *where* $B_1, B_2, ..., B_k$ *are disjoint 2-edge-connected graphs,* $e_i = y_i x_{i+1}$ *is an edge and* x_i, y_i *are distinct vertices of* B_i *for all* $1 \leq i \leq k$. *Let* $H_S = \bigcup_{i=1}^{k} B_i \cup \{e_1, e_2, ..., e_{k-1}\}$.

Let H be a subgraph of G. A *trivial H-bridge* in G is an edge in $E(G)\backslash E(H)$ with both ends in $V(H)$. A *non-trivial H-bridge* in G is a component K of $G\backslash H$ together with all vertices of H adjacent to vertices of K and all edges with one end in H and the other in K. The *vertices of attachment* of a H-bridge B in G are $V(B) \cap V(H)$. The *edges of attachment* of a H-bridge B in G are edges between $V(B)\backslash V(H)$ and $V(B) \cap V(H)$. (See Fig. 2.4.)

A *surface* is a topological space in which every point has an open neighborhood homeomorphic to an open disc in the Euclidean plane. It is a *closed surface* if it is compact.

Give a closed surface S, we can construct two new closed surfaces S_1 and S_2 as follows:

1) Choose a region $D \subseteq S$ which is homeomorphic to a closed disc. Construct S_1 by deleting the interior of D from S and then identifying opposite pairs of points on the boundary of D via the homeomorphism of D to the closed disc.

2) Choose regions $D_1, D_2 \subset F \subset S$ such that D_1 and D_2 are disjoint and there is a homeomorphism f from F to an open disc which maps D_1, D_2 onto closed discs. Construct S_2 by deleting the interior points of D_1 and D_2 from S and then

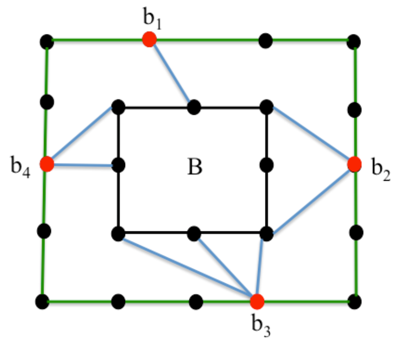


Figure 2.4: The structure of an H-bridge B of a graph G when H is an outer cycle of G. The red vertices are vertices of attachment of B and the blue edges are edges of attachment of B.

identifying the boundaries of D_1 and D_2 in a way that coressponds to a clockwise orientation of $f(D_1)$ in $f(F)$ and a counter-clockwise orientation of $f(D_2)$ in $f(F)$.

We say that S_1 and S_2 are obtained from S by adding a *cross cap* and a *handle*, respectively. The classification of surfaces tells us that every closed surface S is homeomorphic to a surface which can be obtained from the sphere by recursively adding either handles or cross caps. We say that S is an *orientable surface of orientable genus* h if S is homeomorphic to a surface obtained by adding h handles to the sphere, and that S is a *nonorientable surface of nonorientable genus* c if S is homeomorphic to a surface obtained by adding c cross caps to the sphere. The sphere, torus, and double torus are examples of orientable closed surfaces of genus 0, 1 and 2, respectively. The projective plane and Klein bottle are examples of non-orientable closed surfaces of genus 1 and 2, respectively.

We can consider the projective plane as a disc such that opposite points on the boundary are identified. We can also consider the torus as rectangle such that pairs of points on the boundary which lie on a line parallel to one of the sides are identified.

Let G be a connected graph embedded on a surface S. Then every face is bounded by a walk, called a *facial walk*. A facial walk F is called a *facial cycle* if F is a cycle. When G is a plane graph, a facial walk(cycle) of the infinite face of G is called the *outer walk (cycle)*, and is denoted by F_G.

Let C be a cycle in a plane graph G, and $x, y \in V(C)$. Then $C[x, y]$ is the path of C from x to y in the clockwise direction.

A closed curve ϕ on a surface Σ is called *essential* if ϕ can be contracted into one point on Σ. Moreover, a cycle C of a graph G embedded on Σ is called *essential* if the closed curve defined by C is essential. Note that every closed curve in the plane is non-essential, any two essential closed curve in the projective plane must intersect, but this is not necessary true for the torus. (See Fig. 2.5.)

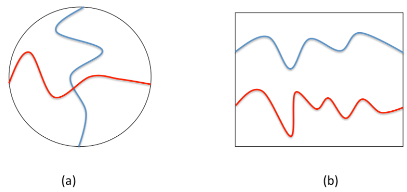


Figure 2.5: Two essential closed curves in the projective plane (a) and the torus (b).

Definition 2.3 *Let R be a face of graph G embedded on surface S, C be the facial walk of R, and $x \in V(C)$. The (x, R)-*width *of G is the smallest number of intersections of G with any essential closed curve T on S which intersect x and R. The R-*width *of G is the smallest (y, R)-width over all $y \in V(C)$. Moreover, the representativity of G on S is the smallest R-width over all faces R of G.*

Definition 2.3 is illustrated in Figure 2.6.

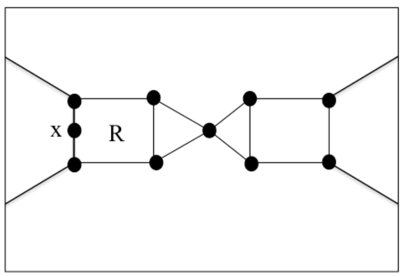


Figure 2.6: A graph embedded on projective plane with representativity 1, R-width 2, and (x, R)-width 3.

The graph G in Fig. 2.7 shows that we cannot always find an F_G-Tutte trail of G from x to y containing two given edges on F_G. However, G does have a Tutte trail T from x to y containing e_1 and e_2 such that H is the only T-bridge of G containing a vertex of F_G. Hence $G\backslash(H\backslash\{x, y, z\})$ has a $C\backslash(H\backslash\{x, y, z\})$-Tutte trail from x to y containing e_1 and e_2. We will use the idea of this subgraph H in the following definition.

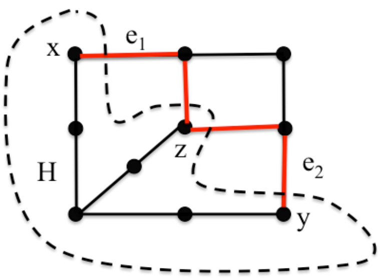


Figure 2.7: The structure of Tutte trail T of a graph G.

Definition 2.4 *Let G be a connected graph embedded on a surface S. Let R be a face of G and C be a subwalk of the facial walk of R. A C-*flap *in G is either a single vertex $\{a\}$ for some vertex $a \in V(C)$, or an $\{a, b, c\}$-bridge H of G for some $a, b, c \in V(G)$ such that*
(i) $a, b \in V(C) \cap V(H)$ and $a \neq b$.
(ii) H contains a subwalk C' of C from a to b.
(iii) H is a planar graph and C' and c are on the outer walk of H.
*We say that H is trivial if $|V(H)| = 1$. Moreover, H is a C-*edge-flap *of G if H is either trivial, or a, b and c have degree one in H.*

Definition 2.4 is illustrated in Figure 2.8.

Throughout the thesis, we will frequently divide long proofs into several claims. We will denote the end of a proof of a theorem or lemma by ■ and the end of the proof of a claim by □.

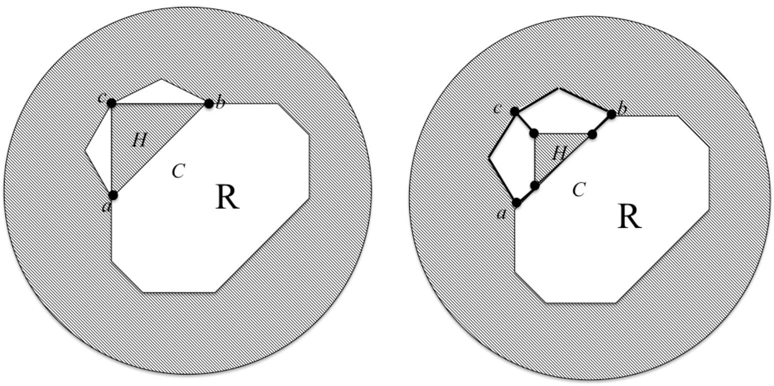


Figure 2.8: The structure of a C-flap (left) and a C-edge-flap (right) in G where C is the facial walk of the face R.

Chapter 3

Tutte Subgraphs of Plane Graphs

In this chapter, the main result (Theorem 3.1(a)) is that every 2-edge-connected plane graph G has an F_G-Tutte trail from u to v containing e for any $u, v \in V(G)$ and $e \in E(F_G)$. For any facial walk F of a graph G, we can redraw G such that F is an outer walk of the new graph. Hence we can restate the main result as every 2-edge-connected plane graph G has an F-Tutte trail from u to v containing e for any facial walk F, $u, v \in V(G)$ and $e \in E(F)$.

In Section 3.1, we prove the main result and its corollary. In Section 3.2, we prove other results on Tutte subgraphs of plane graphs. These will be used in Chapter 4 and Chapter 5.

3.1 Tutte trails of plane graphs

This section contains the main result which will be used throughout this thesis.

We use ideas from [32], [28], and [29] to prove the following theorem.

Theorem 3.1 *Let G be a 2-edge-connected plane graph and F_G be the outer walk of G.*
(a) If $u, v \in V(G)$ and $e \in E(F_G)$, then there is an F_G-Tutte trail in G from u to v containing e.
(b) If $|E(F_G)| \geq 3$ and $e_1, e_2, e_3 \in E(F_G)$, then there is an F_G-Tutte closed trail in G containing e_1, e_2 and e_3.
(c) If $E(F_G) = \{e_1, e_2\}$, then there is an F_G-Tutte closed trail in G containing e_1

and e_2.

Proof. We proceed by contradiction. Suppose the theorem is false and choose a counterexample G such that $|V(G)|$ is small as possible and, subject to this condition, $|E(G)|$ is as small as possible. It can checked that the theorem is true when $|V(G)| \leq 3$. So we have $|V(G)| \geq 4$.

Claim 1 *(c) holds for* G.

Proof of Claim 1. Let $V(F_G) = \{x_1, x_2\}$. If $G - e_1$ is 2-edge-connected, then by induction on (a), $G - e_1$ has an F_{G-e_1}-Tutte trail T' from x_1 to x_2 containing e_2. Thus $T' \cup \{e_1\}$ is an F_G-Tutte closed trail containing e_1 and e_2 in G. Hence we may assume that $G - e_1$ is not 2-edge-connected. Then $G - \{e_1, e_2\} = B_1 \cup B_2$ where B_i are edge-blocks containing x_i. (Possibly, $V(B_i) = \{x_i\}$.) By induction on (a), B_i has an F_{B_i}-Tutte trail T_i from x_i to x_i. (Possibly, $T_i = \{x_i\}$.) Thus $T_1 \cup T_2 \cup \{e_1, e_2\}$ is a Tutte closed trail containing e_1 and e_2. $\square$

Next, we divide the proof into three cases and prove (a) and (b) simultaneously for each case. Claim 2 and Claim 3 are the results of (a) and (b), respectively, in Case 1. Claim 4 and Claim 5 are the results of (a) and (b), respectively, in Case 2. Claim 6 and Claim 7 together are the results of (a) in Case 3, and Claim 8 is the result of (b) in Case 3.

Case 1: G has a 1-separation (G_1, G_2).
Let $V(G_1) \cap V(G_2) = \{x\}$. (See Fig. 3.1 and 3.2.)

Claim 2 *(a) is true for* G *in Case 1.*

Proof of Claim 2. Assume without loss of generality that $e \in E(G_1)$.

Suppose $u, v \in V(G_1)$. Then, by induction on (a), G_1 has an F_{G_1}-Tutte trail T_1 from v to u containing e. If $x \notin V(T_1)$, then T_1 is an F_G-Tutte trail from u to v containing e in G. If T_1 contains x, then G_2 has, by induction on (a), an F_{G_2}-Tutte trail T_2 from x to x and $T_1 \cup T_2$ is an F_G-Tutte trail from u to v containing e in G

Suppose $v \in V(G_1)$ and $u \in V(G_2)$. Then, by induction on (a), G_1 has an F_{G_1}-Tutte trail T_1' from v to x containing e and G_2 has an F_{G_2}-Tutte trail T_2' from u to x. Then $T_1' \cup T_2'$ is an F_G-Tutte trail from u to v containing e in G.

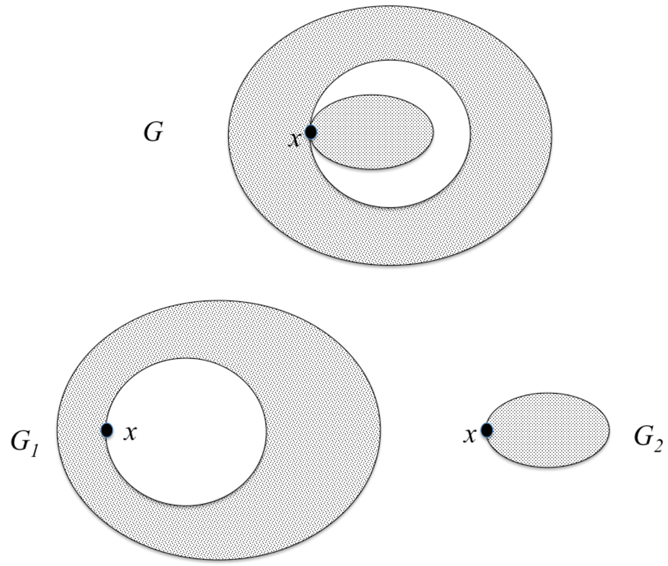


Figure 3.1: the graphs G, G_1 and G_2 when $x \notin V(F_G)$.

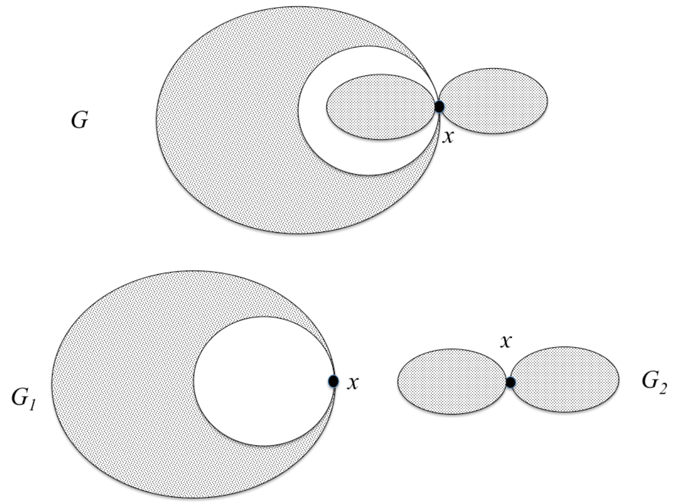


Figure 3.2: the graphs G, G_1 and G_2 when $x \in V(F_G)$.

Suppose $u, v \in V(G_2)$. Then, by induction on (a), G_1 has an F_{G_1}-Tutte trail T_1'' from x to x containing e and G_2 has an F_{G_2}-Tutte trail T_2'' from u to v containing an edge in F_{G_2} incident to x. Then $T_1'' \cup T_2''$ is an F_G-Tutte trail from u to v containing e in G. $\square$

Claim 3 *(b) is true for G in Case 1.*

Proof of Claim 3. Assume without loss of generality that $e_1, e_2 \in E(G_1)$.

Suppose $e_3 \in E(G_1)$. Then, by induction on (b), G_1 has an F_{G_1}-Tutte closed trail T_1 containing e_1, e_2 and e_3. If $x \notin V(T_1)$, then T_1 is an F_G-Tutte closed trail containing e_1, e_2 and e_3 in G. If T_1 contains x, then G_2 has, by induction on (a), an F_{G_2}-Tutte trail T_2 from x to x and $T_1 \cup T_2$ is an F_G-Tutte closed trail containing e_1, e_2 and e_3 in G.

Suppose $e_3 \in E(G_2)$. Then, by induction on (a), G_2 has an F_{G_2}-Tutte closed trail T_2' from x to x containing e_3. If both e_1 and e_2 are incident to x, then, by induction on (b) or (c), G_1 has an F_{G_1}-Tutte closed trail T_1' containing e_1 and e_2 and $T_1' \cup T_2'$ is an F_G-Tutte closed trail containing e_1, e_2 and e_3 in G. Assume without loss of generality that e_2 is not incident to x. Then, by induction on (b), G_1 has an F_{G_1}-Tutte closed trail T_1'' containing e_1, e_2 and an edge in $G_1 \cap F_G$ incident to x. Thus $T_1'' \cup T_2'$ is an F_G-Tutte closed trail containing e_1, e_2 and e_3 in G. □

Hence Case 1 cannot occur and thus G is 2-connected.

Case 2 : G has a 2-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{x, y\}$ and $x, y \in V(F_G)$. (See Fig. 3.3.) We assume further that:
(i) if (a) is false for G and $v \in V(F_G)$, then $v \in V(G_1)$, $e \in E(G_2)$, and $V(G_1)\backslash\{v, x, y\} \neq \emptyset$;
(ii) if (a) is false for G and $u, v \notin V(F_G)$, then $v \in V(G_1)$ and $e \in E(G_2)$;
(iii) if (b) is false for G, then $e_1 \in E(G_1)$ and $e_3 \in E(G_2)$.

In these cases we choose G_1 and G_2 such that $|V(G_2)|$ is as small as possible, subject to satisfying the hypotheses of Case 2.

Let G_i' be the graph obtained from G_i by adding an edge f_i between x and y such that $(F_{G_i} - f_i) \subseteq F_G$. (See Fig. 3.4.) We let G_2'' be the graph obtained from G_2 by adding a vertex w and joining it to x and y such that $(F_{G_2''}\backslash\{w\}) \subseteq F_G$. (See Fig. 3.4.) Then G_1', G_2' and G_2'' are 2-connected.

Claim 4 *(a) is true for G in Case 2.*

Proof of Claim 4. Suppose $u \in V(G_1)$. Then, by induction on (a), G_1' has an $F_{G_1'}$-Tutte trail T_1 from u to v containing f_1. By induction on (b) or (c), G_2' has an $F_{G_2'}$-Tutte closed trail T_2 containing e and f_2. Then $(T_1 - f_1) \cup (T_2 - f_2)$ is a desired F_G-Tutte trail from u to v.

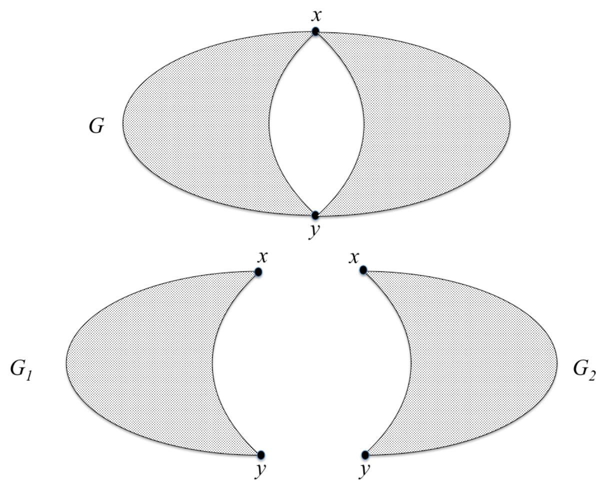


Figure 3.3: the graphs G, G_1, G_2.

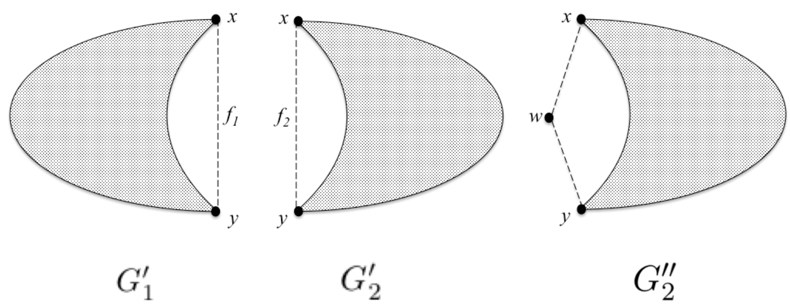


Figure 3.4: the graphs G_1', G_2' and G_2''.

Suppose $u \in V(G_2)\backslash V(G_1)$ and $v \in \{x, y\}$. Assume without loss of generality that $v = x$. By induction on (a), G_2' has an $F_{G_2'}$-Tutte trail T_2' from v to u containing e. If T_2' does not contain y, then, since y lies on the outer face of G_2', the component of $G_2'\backslash V(T_2')$ containing y has two edges connecting it to v and to another vertex z of T_2'. Thus $\{v, z\}$ separates v from e and u which contradicts the minimality property of G_2. Hence we may assume that T_2' contains y. If $f_2 \in E(T_2')$, then, by induction on (a), G_1' has an $F_{G_1'}$-Tutte trail T_1' from v to v containing f_1 and $(T_1' - f_1) \cup (T_2' - f_2)$ is an F_G-Tutte trail from v to u containing e in G. If $f_2 \notin E(T_2')$, then, by induction on (a), G_1' has an $F_{G_1'}$-Tutte trail T_1'' from v to y containing f_1 and $(T_1'' - f_1) \cup T_2'$ is an F_G-Tutte trail from v to u containing e in G.

Suppose $u \in V(G_2)\backslash V(G_1)$ and $v \in V(G_1)\backslash\{x, y\}$. Then, by induction on (a), G_2'' has an $F_{G_2''}$-Tutte trail T_2'' from w to u containing e. Assume without loss of generality that $wx \in V(T_2'')$. If T_2'' does not contain y, then, since y lies on the outer face of G_2'', the component of $G_2''\backslash V(T_2'')$ containing y has two edges connecting it to w and to another vertex z of T_2''. Thus $\{x, z\}$ separates v from e and u which contradicts the minimality property of G_2. Hence we may assume that T_2'' contains y. By induction on (a), G_1' has an $F_{G_1'}$-Tutte trail T_1 from v to y containing f_1. Thus $(T_1 - f_1) \cup (T_2'' - w)$ is a desired F_G-Tutte trail from u to v containing e in G. $\square$

Claim 5 *(b) is true for G in Case 2.*

Proof of Claim 5. Assume without loss of generality that $e_2 \in E(G_1)$. By induction on (b), G_1' has an $F_{G_1'}$-Tutte closed trail T_1 containing e_1, e_2 and f_1. By induction on (b) or (c), G_2' has an $F_{G_2'}$-Tutte closed trail T_2 containing e_3 and f_2. Then $(T_1 - f_1) \cup (T_2 - f_2)$ is an F_G-Tutte closed trail containing e_1, e_2 and e_3. $\square$

Hence Case 2 cannot occur. By symmetry, we may claim that G does not exist a 2-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{x, y\}$, $x, y \in V(F_G)$, $u \in V(G_1)$, $e \in E(G_2)$ and, if $u \in V(F_G)$, then $V(G_1)\backslash\{u, x, y\} \neq \emptyset$

Case 3 : G does not have a 2-separation in the type of Case 2.
We will prove three claims as follows.

- **Claim 6**: We consider (a) when $v \in V(F_G)$. Then G does not have a 2-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{x, y\}$, $x, y \in V(F_G)$, $v \in V(G_1)$, $e \in E(G_2)$ and $V(G_1)\backslash\{v, x, y\} \neq \emptyset$.
- **Claim 7**: We consider (a) when $u, v \notin V(F_G)$. Then G does not have a 2-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{x, y\}$, $x, y \in V(F_G)$, $v \in V(G_1)$ and $e \in E(G_2)$.
- **Claim 8**: We consider (b). Then G does not have a 2-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = \{x, y\}$, $x, y \in V(F_G)$, $e_1 \in E(G_1)$ and $e_3 \in E(G_2)$.

Next, we give some definitions which we use in this case. Let $F \subseteq F_G$ and $H \subseteq G\backslash V(F)$. Let X be an $(F \cup H)$-bridge. If X has vertices of attachment in F, we let Q_X be the minimal path in F including all vertices of attachment of X in F such that no interior vertex of Q_X shares a face with a vertex of H. (Possibly, Q_X is a single vertex.) Let p_X and q_X be the end vertices of Q_X. (Possibly, $p_X = q_X$.) Note that for two $(F \cup H)$-bridges X, Y, either $Q_X \subset Q_Y$, $Q_Y \subset Q_X$, or $E(Q_X) \cap E(Q_Y) = \emptyset$. An $(F \cup H)$-bridge X is *maximal* if $|V(Q_X)| = 1$, or there is no $(F \cup H)$-bridge Y such that $Q_X \subset Q_Y$.

An $(F \cup H)$-*bridge group* A is the union of a maximal $(F \cup H)$-bridge X together with all $(F \cup H)$-bridges Y such that $|V(Q_Y)| \geq 2$ and $Q_Y \subset Q_X$. We put $Q_A = Q_X$, $p_A = p_X$ and $q_A = q_X$ where X is the maximal $(F \cup H)$-bridge in A. An $(F \cup H)$-bridge group is called *trivial* if it has only two vertices.

An (F, H)-*connector* in G is a bridge of $F \cup H$ in G with its vertices of attachment in both F and H. An $(F \cup H)$-connector is called *trivial* if it has only two vertices. An (F, H)-*connector group* L is an $(F \cup H)$-bridge group which contains an (F, H)-connector in G. An $(F \cup H)$-connector group is called *trivial* if it has only two vertices.

Claim 6 *If either u or v are on F_G, (a) is true for G in Case 3.*

Proof of Claim 6. Assume without loss of generality that $v \in V(F_G)$. Let P_1 be a path of F_G from v to an end vertex of e such that $e \notin E(P_1)$ and $u \notin P_1 \backslash \{v\}$ ($P_1 = v$ if v is an end vertex of e). Assume without loss of generality that P_1 traverses F_G in a clockwise direction as we move from v to e. Let w be the adjacent vertex of v on F_G which is not in P_1. If $u = v$, put $u_1 = w$ and otherwise, put $u_1 = u$. Let v_1 be the end of e which is not on P_1 and let $P_2 = F_G \backslash V(P_1)$ and $H = G \backslash V(P_1)$.
Suppose $|V(F_G)| \geq 3$ and v_1 has no neighbor in $G \backslash V(F_G)$, then the two neighbors of v_1 on F_G separate G as in Claim 4 of Case 2 since $|V(G)| \geq 4$. This contradicts the assumption. Thus v_1 must have at least one neighbors in $G \backslash V(F_G)$. Next, we will show that there is a unique block B of H containing P_2.

We first consider the case when $V(P_2) = \{v_1\}$. Let B be a block of H containing v_1. If v_1 has only one neighbor in H, then B is a only block of H containing v_1 and its neighbor in H. (See Fig. 3.5.) Hence we may suppose v_1 has at least two neighbors in H. Let x and y be distinct neighbors of v_1 in H. Assume x and y belong to different blocks of H containing v_1 such that $v_1 x$ lies on the same face of G as $v_1 v$. Let C_x be the block of H containing x. Since G is 2-connected, we can let

r be the vertex on P_1 which is connected to C_x and has the largest distance on P_1 from v. (Possibly, $r = v$.) (See Fig. 3.6.) Then G has a 2-separation of the type in Claim 4 of Case 2 by using v_1 and r which is a contradiction. Hence we may assume that all neighbors of v_1 in H are in the same block of H containing v_1. Then B is the only block of H containing v_1.

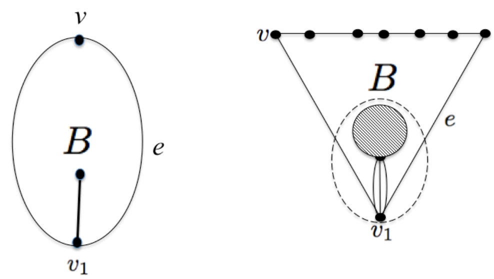


Figure 3.5: The block B in the case v_1 has one neighbor in H.

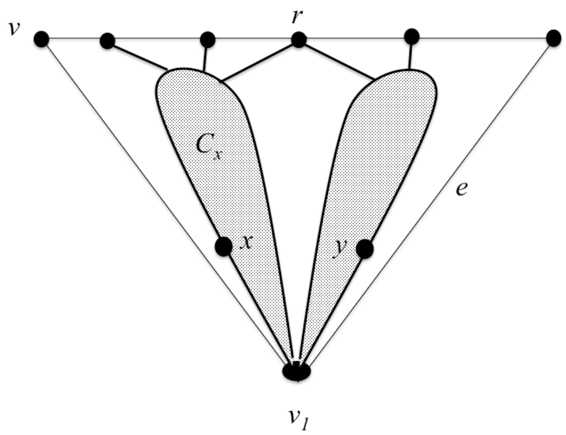


Figure 3.6: The structure of G when x and y are in different components of $G\backslash V(F)$.

We next consider the case when $V(P_2) = \{v_1, w\}$. If there is no v_1w-path P in $G - v_1w$ disjoint from $V(P_1)$, then Menger's theorem and planarity imply that there is a vertex $x \in V(P_1)$ such that every v_1w-path must contain x. This is a contradiction because G will have a 2-separation of the type of Claim 4 in Case 2 by using $\{x, w\}$ or $\{x, v_1\}$.

We next consider the case when $|V(P_2)| \geq 3$. Suppose that there is no v_1w-path in G disjoint from $V(F_G)\backslash\{v_1, w\}$, then Menger's theorem and planarity imply that there are vertices $x \in V(P_1)$ and $y \in V(P_2)\backslash\{v_1, w\}$ such that every v_1w-path must contain at least one of these vertices. This is a contradiction because G will have a 2-separation of the type in Case 2 by using x and y. Hence we may suppose that there is a v_1w-path P disjoint from $V(F_G)\backslash\{v_1, w\}$. Let B be the block of H containing $P \cup P_2$. Then B is the only block of H containing P_2.

Now, we define B' as the edge-block of H containing B when $B \neq K_2$. (Possibly, $B' = B$.) If $B = K_2$, we define $B' = \{v_1\}$. (See Fig. 3.7.)

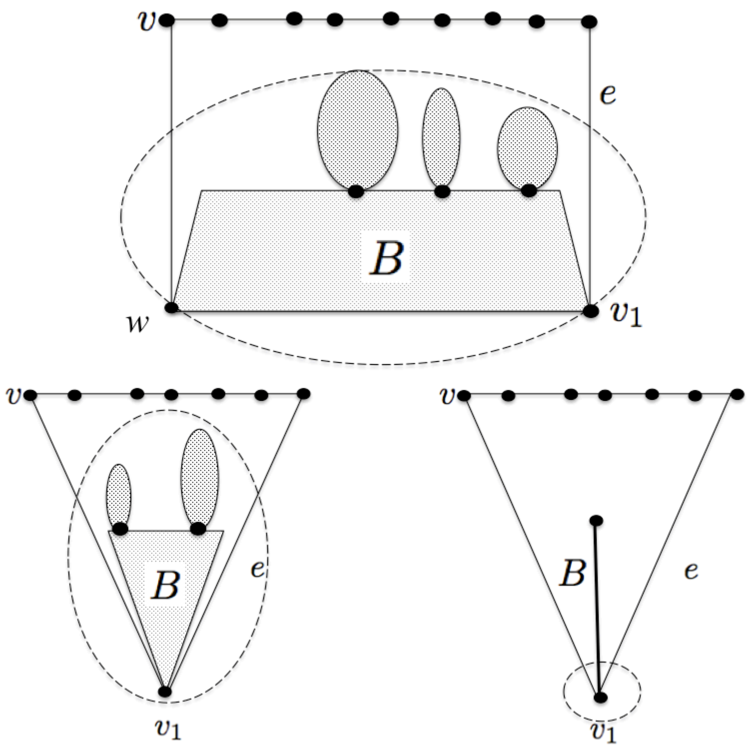


Figure 3.7: The structure of B'.

We can assume u_1 is in the connected component of H containing P_2 since otherwise we could interchange P_1 and P_2. If $u_1 \notin B'$, then there is a unique vertex u' of B' such that each path in H from u_1 to B' contains u'. (See Fig. 3.8.) If $u_1 \in B'$, let $u_1 = u'$.

If $B \neq K_2$, then B' is 2-edge-connected and by induction on (a), we can choose an $F_{B'}$-Tutte trail T' from v_1 to u' containing an edge of $F_{B'}$ incident to w in B'. If

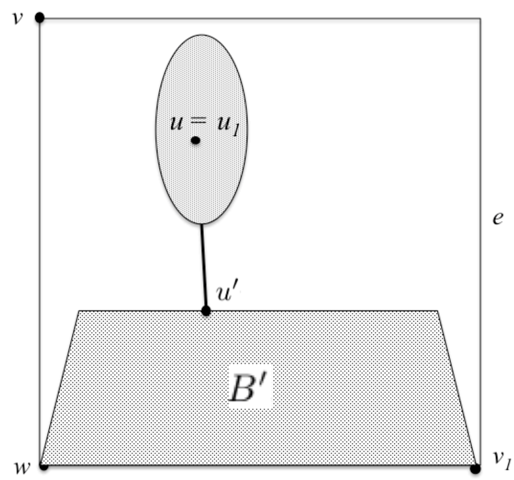


Figure 3.8: Vertices u_1 and u' in the case $u_1 \notin V(B')$.

$B = K_2$, put $T' = \{v_1\}$. Then $T = P_1 \cup T' \cup \{e\}$ is a vu'-trail. We will modify T by diverting it into each $(P_1 \cup T')$-bridge group J of G such that J has vertices of attachment on P_1 to obtain the desired F_G-Tutte trail from u to v containing e.

Consider any $(P_1 \cup T')$-bridge group J of G such that J has vertices of attachment on P_1. By construction of B' and T', J has at most two edges of attachment on T'. If $u_1 \notin V(B')$, then we let J'' be the $(P_1 \cup T')$-bridge group of G such that $u \in V(J)$. Next, we let J be a $(P_1 \cup T')$-bridge group of G such that $J \neq J''$, J has vertices of attachment on P_1, and J has more than three edges of attachment on $P_1 \cup T'$. We consider J in three subcases as follows. (See Fig. 3.9.)

Subcase 3.1: J has no edge of attachment on T'.
Then, by induction on (a), $J \cup Q_J$ has a $p_J q_J$-Tutte trail T_J from p_J to q_J. In T, we replace Q_J by T_J for each such J.

Subcase 3.2: J has one edge of attachment on T'.
Let v_J be the vertex of attachment of J in T' and J^* be the union of J, Q_J and the new edge $e_J = p_J v_J$. (See Fig. 3.10.) Then J^* is 2-edge-connected and by induction on (a), has an F_{J^*}-Tutte trail T_J from q_J to v_J containing e_J. In T, we replace Q_J by $T_J - v_J$ for each such J.

Subcase 3.3: J has two edges of attachment on T'.
Let a_1, b_1 be vertices of attachment of J on T' (possibly, $a_1 = b_1$) and let a_2 and

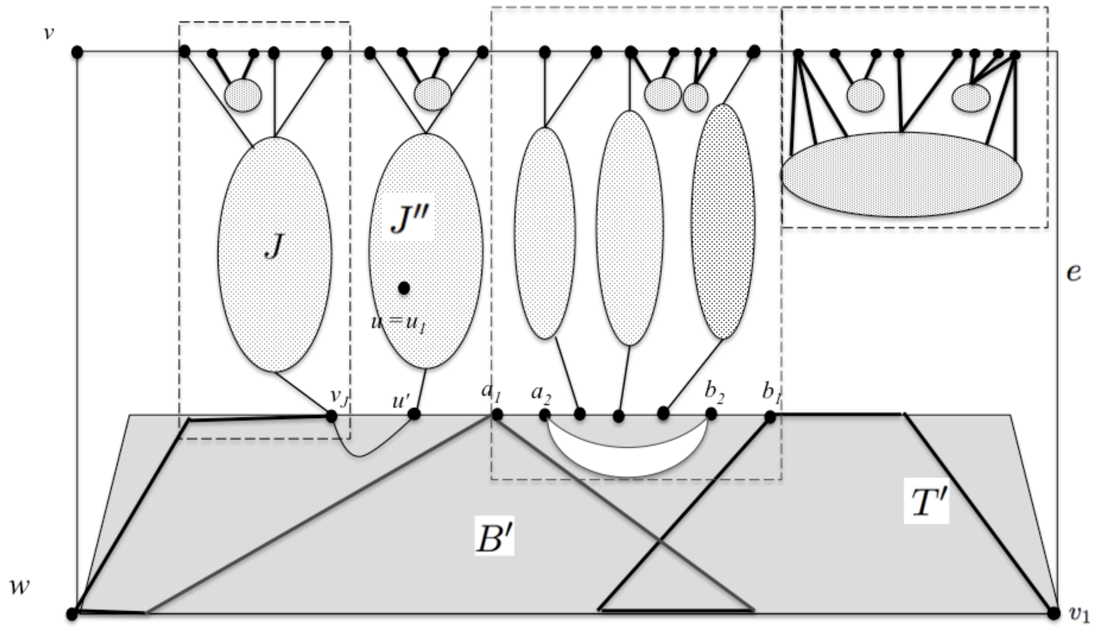


Figure 3.9: $(P_1 \cup T')$-bridge groups of G.

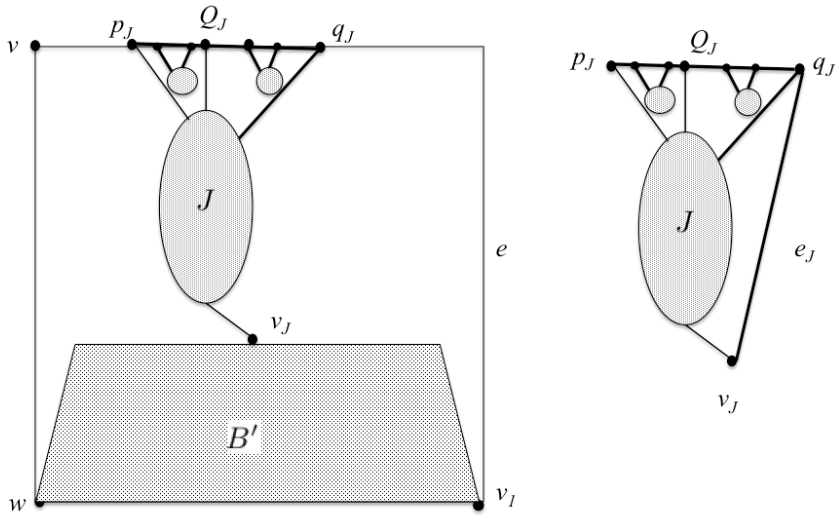


Figure 3.10: The structure of J^*.

b_2 be the vertices in J which are adjacent to a_1 and b_1, respectively. (Possibly, $a_2 = b_2$.) (See Fig. 3.11.) Let $J_1 = (J \cup Q_J)\backslash\{a_1, b_1\}$ and J_2 be the edge-block of J_1 containing Q_J. We will consider two subcases.

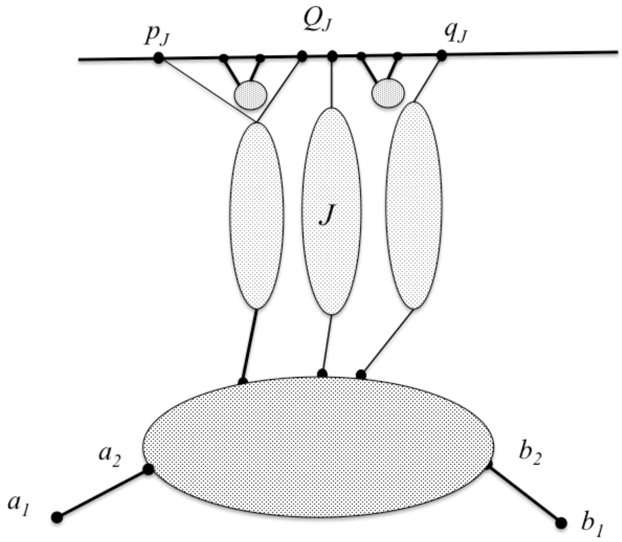


Figure 3.11: The structure of J in Subcase 3.3.

Subcase 3.3.1: $J_1 = J_2$.
Then, by induction on (a), J_2 has an F_{J_2}-Tutte trail T_1 from p_J to q_J containing an edge of F_{J_2} incident to a_2. In T, we replace Q_J by T_1. Note that if $b_2 \notin T_1$, the component S of $J_1 - T_1$ containing b_2 has exactly two edges connecting it to T_1 and then has exactly three edges connecting it to T.

Subcase 3.3.2: $J_1 \neq J_2$.
Then there are components A and B of $J_1 - J_2$ (possibly, $A = B$) such that $a_2 \in V(A)$ and $b_2 \in V(B)$. Since J_2 is an edge-block of J_1, both A and B have only one edge connecting them to J_2. Let a_3 and b_3 be vertices in J_2 which are adjacent to A and B, respectively. (See Fig. 3.12.) By induction on (a), J_2 has an F_{J_2}-Tutte trail T_2 from p_J to q_J containing an edge of F_{J_2} incident to a_3. In T, we replace Q_J by T_2. Note that if $b_3 \notin T_2$, the component S' of $J_1 - T_2$ containing b_3 has exactly two edges connecting it to T_2 and the union of S', B and the edge between S' and B is

a component of $G\backslash V(T)$ which has exactly three edges connecting it to T.

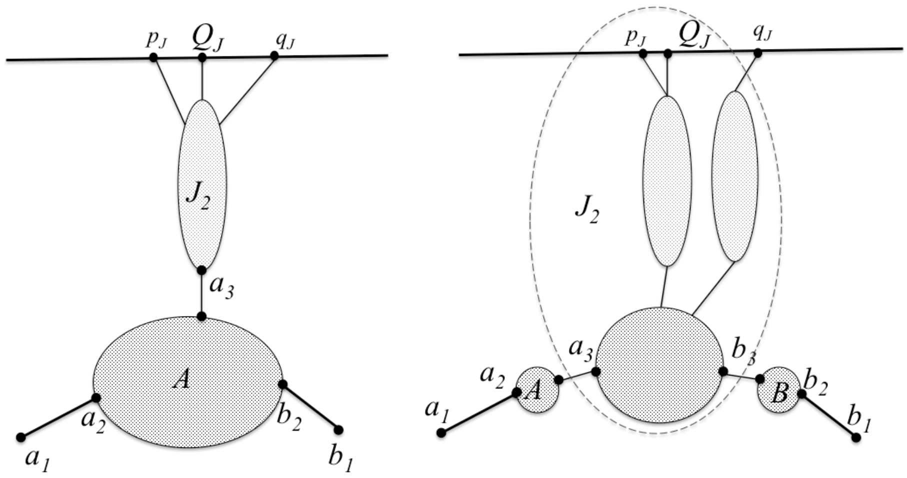


Figure 3.12: The structure of J when $J_1 \neq J_2$.

We have constructed an F_G-Tutte trail T from v to u' containing e. If $u' = u_1 = u$, then T is the desired Tutte trail in G. If $u = v(u' = u_1 = w)$, then $T \cup \{vw\}$ is the desired Tutte trail. So suppose $u \neq u'$. Then $u \in V(J'')$. (See Fig. 3.9.) Let K be the graph obtained from the union of $J'', Q_{J''}$ and the new edge $e_K = q_J u'$. Then K is 2-edge-connected and, by induction on (a), has an F_K-Tutte trail T_K from p_J to u containing e_K. Hence $(T - Q_{J''}) \cup (T_K - e_K)$ is a desired F_G-Tutte trail from u to v containing e. □

Claim 7 *If $u, v \notin V(F_G)$, then (a) is true for G in Case 3.*

Proof of Claim 7. Then u and v are in the same component of $G\backslash V(F_G)$. According to the edge-block tree of $G\backslash V(F_G)$, there is a unique path of edge-blocks H of $G\backslash V(F_G)$ such that $V(H) = \bigcup_{i=1}^{k} V(B_i)$ where B_i is an edge-block in $G\backslash V(F_G)$, $u = a_1 \in V(B_1), v = b_k \in V(B_k)$ and $E(H) = \bigcup_{i=1}^{k} E(B_i) \cup \{b_i a_{i+1} \in E(G) | b_i \in V(B_i), a_{i+1} \in V(B_{i+1}), 1 \leq i \leq k-1\}$. (See Fig. 3.13.)

For each (F_G, H)-connector group M of G, let v_M be the unique vertex of attachment of M in H and i_M be the integer such that $v_M \in B_{i_M}$. Let K be the (F_G, H)-connector group such that $e \in E(Q_K)$, or if there is no such group, the

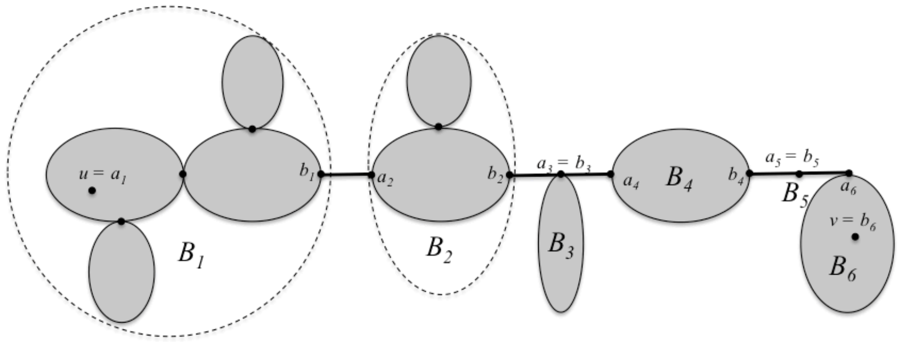


Figure 3.13: The structure of H.

(F_G, H)-connector group with q_K nearest to e counterclockwise from it, and such that K shares a face with either the edge e or the $(F_G \cup H)$-bridge group A with $e \in E(Q_A)$. (See Fig. 3.14.)

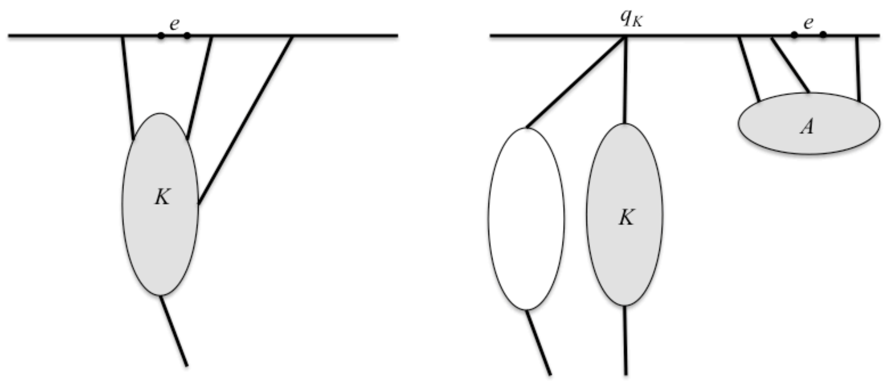


Figure 3.14: The structure of K.

Let L be the (F_G, H)-connector group such that q_L is nearest counterclockwise to p_K (possibly, $q_L = p_K$) and q_L, v_L, p_K, e_K all lie on the same face of G, where e_K is the edge of K containing v_K. (See Fig. 3.15 and 3.16.) Let P_1 and P_2 be the paths in F_G from p_L counterclockwise to q_K and the path in F_G from q_L clockwise to p_K, respectively. (Possibly, $P_2 = \{p_K\}$.)

Let $f = \min\{i_K, i_L\}, g = \max\{i_K, i_L\}$. Next, we will construct F_{B_i}-Tutte trails (or a pair of trails in Subcase 3.5) T_i of B_i for $i \in \{f, g\}$ as follows.

Subcase 3.4: $f = g$ and $v_K = v_L$.

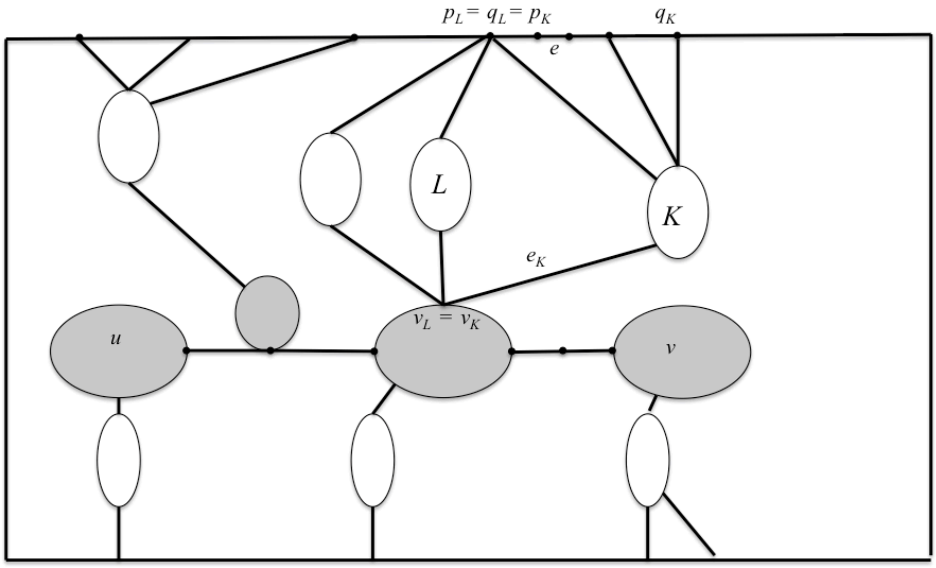


Figure 3.15: The structure of K and L when $p_K = q_L$.

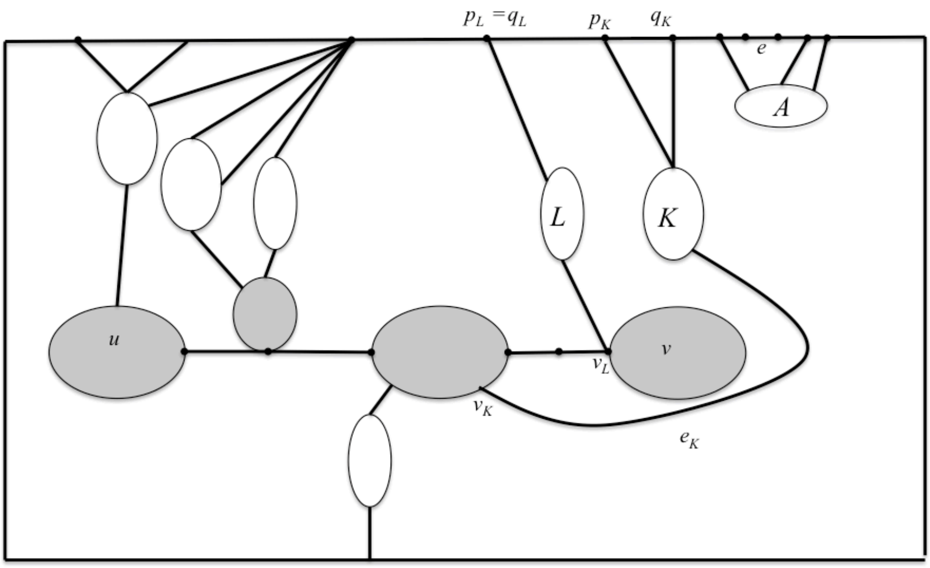


Figure 3.16: The structure of K and L when $p_K \neq q_L$.

Then, by induction on (a), B_f has an F_{B_f}-Tutte trail T_f from a_f to b_f containing v_K.

Subcase 3.5: $f = g$ but $v_K \neq v_L$.
We let $R = B_f \cup \{v_K v_L\}$ with the new edge embedded in the face shared between K and L. (See Fig. 3.17.) By induction on (a), R has an F_{B_R}-Tutte trail T'_f from a_f to b_f containing $v_K v_L$. Let $T_f = T'_f - \{v_K v_L\}$.

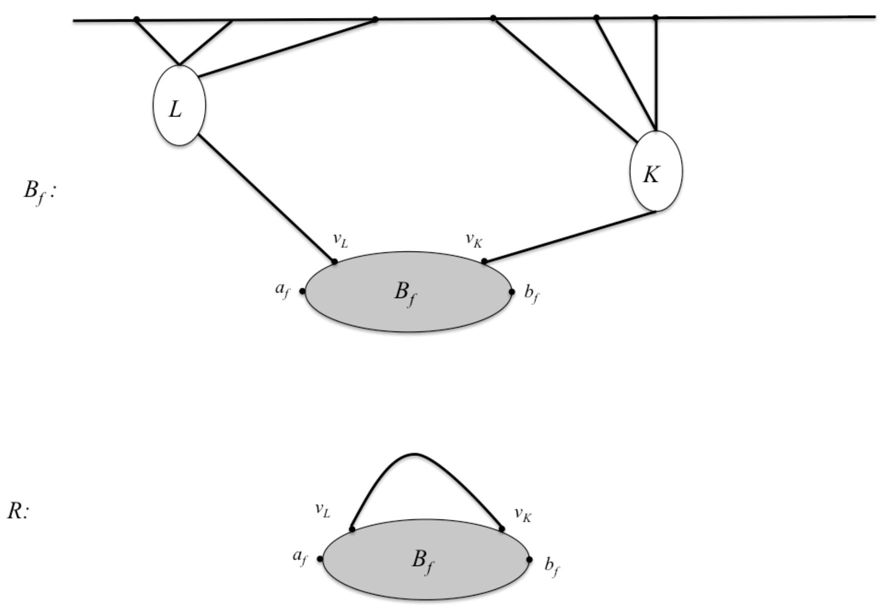


Figure 3.17: The structure of R and B_f when $v_K, v_L \in V(B_f)$ but $v_K \neq v_L$.

Subcase 3.6: $f \neq g$.
We let $z_1, z_2 \in \{v_K, v_L\}$ such that $z_1 \in V(B_f)$ and $z_2 \in V(B_g)$. By induction on (a), B_f has an F_{B_f}-Tutte trail T_f from a_f to z_1 containing an edge of F_{B_f} incident to b_f, and B_g has an F_{B_g}-Tutte trail T_g from b_g to z_2 containing an edge of F_{B_g} incident to a_g.

For $1 \leq i < f$ or $g < i \leq k$, by induction on (a), B_i has an F_{B_i}-Tutte trail T_i from a_i to b_i in B_i (if $B_i = \{a_i\}$, we let $T_i = \{a_i\}$).

Next, we will define a Tutte trail T_K of K as follows. If K is a trivial (F_G, H)-connector group, we let $T_K = K$. For the case when K is a non-trivial (F_G, H)-connector group, we let $e_1 = v_K q_K$ and $M = K \cup Q_K \cup \{e_1\}$ with e_1 embedded in the outer walk such that $V(Q_K) \subset V(F_M)$ and we let d be an edge of F_M containing

p_K. (See Fig. 3.18.) If $e \notin E(Q_K)$, then, by induction on (b) or (c), M has an F_M-Tutte closed trail C containing d and e_1. For the case $e \in E(Q_K)$, by induction on (b), M has an F_M-Tutte closed trail C containing d, e_1 and e. (Possibly, $d = e$.) In both case, we let $T_K = C - e_1$.

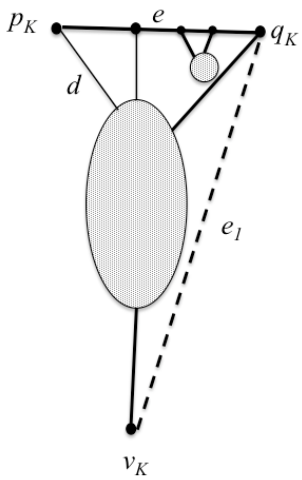


Figure 3.18: The structure of M when $e \in V(Q_K)$.

Next, we will define a Tutte trail T_L of L in the similar way to T_K. If L is a trivial (F_G, H)-connector group, we let $T_L = L$. For the case when L is a non-trivial (F_G, H)-connector group, we let $e_2 = v_L p_L$ and $N = L \cup Q_L \cup \{e_2\}$ with e_2 embedded in the outer walk such that $V(Q_L) \subset V(F_N)$ and we let d' be an edge of F_N containing q_L. By induction on (b) or (c), N has an F_N-Tutte closed trail C' containing d' and e_2. Let $T_L = C' - e_2$.

Finally, we let

$$T' = (\bigcup_{i=1}^{f} T_i) \cup (\bigcup_{i=1}^{f-1} b_i a_{i+1}) \cup (\bigcup_{i=g}^{k} T_i) \cup (\bigcup_{i=g}^{k-1} b_i a_{i+1})$$

and $T = T' \cup T_K \cup T_L \cup P_1$. (See Fig. 3.19.)

Next, we will modify T by diverting it into the $(P_1 \cup T')$-bridge groups J of G such that J has vertices of attachment on P_1 to obtain the desired F_G-Tutte trail from u to v containing e.

Let J be a $(P_1 \cup T')$-bridge group of G such that $J \neq K$, $J \neq L$, J has vertices

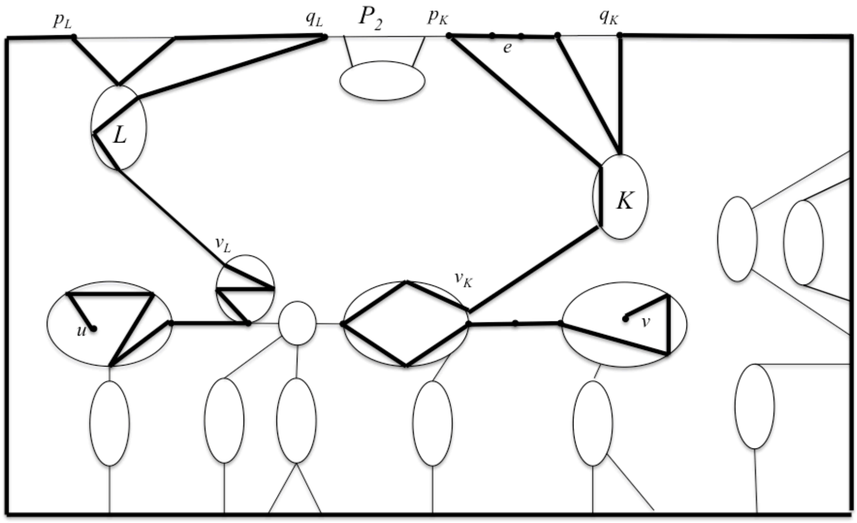


Figure 3.19: The structure of T.

of attachment on P_1, and J has more than three edges of attachment on $P_1 \cup T'$. By construction of H and T', J has at most two vertices of attachment on T'. Then we can modify J in three cases, depending on the number of vertices of attachment on T' using the same method as Subcase 3.1, 3.2, 3.3 of Claim 6.

We have now constructed an F_G-trail T from u to v containing e. If P_2 is a single vertex, then T is the desired F_G-Tutte trail in G. So suppose $V(P_2) \geq 2$ and let J' be the T-bridge in G containing P_2, and J^* be the graph obtained from J' by adding an edge e_{J^*} between q_L and p_K such that $(F_{J^*} - e_{J^*}) = P_2$. Then, by induction on (a), J^* has F_{J^*}-Tutte trail T_{J^*} from q_L to p_K containing e_{J^*}. Hence $T \cup (T_{J^*} - e_{J^*})$ is the desired F_G-Tutte trail from u to v containing e in G. $\square$

Claim 8 *(b) is true for G in Case 3.*

Proof of Claim 8. Let $e_1 = x_1x_2, e_2 = y_1y_2$ and $e_3 = z_1z_2$. Assume without loss of generality that the vertex sequence $x_1, x_2, y_1, y_2, z_1, z_2$ is in clockwise order around F_G.

Let P_1 be the segment of F_G from x_2 clockwise to y_1. (Possibly, $P_1 = x_2 = y_1$.) Let P_2 be the segment of F_G from x_1 counter-clockwise to y_2. Note that $|V(P_2)| \geq 2$ and $e_3 \in E(P_2)$.

Suppose $|V(P_2)| = 2$ and there is no x_1y_2-path in $G - e_3$ disjoint from $V(P_1)$, then Menger's theorem and planarity imply there is a vertex $x \in V(P_1)$ such that every x_1y_2-path must contain x. This is a contradiction because G will have a separation of the type in Case 2 by using $\{x, x_1\}$ or $\{x, y_2\}$.

We next consider the case when $|V(P_2)| \geq 3$. If there is no x_1y_2-path in G disjoint from $V(F_G)\backslash\{x_1, y_2\}$. Then Menger's theorem and planarity imply there are vertices $x \in V(P_1)$ and $y \in V(P_2)\backslash\{x_1, y_2\}$ such that every x_1y_2-path must contain at least one of these vertices. This is a contradiction because G will have a separation of the type in Case 2 by using $\{x, y\}$.
Hence in both cases we may suppose that there is a x_1y_2-path P disjoint from $V(F_G)\backslash\{x_1, y_2\}$. Since $P \cup P_2$ is 2-connected, there is a unique block B of $G\backslash V(P_1)$ containing P_2.

Let B' be the edge-block of $G\backslash V(P_1)$ containing B. (See Fig. 3.20.) By induction on (a), B' has an $F_{B'}$-Tutte trail T' from x_1 to y_2 containing e_3. Let $T = T' \cup P_1 \cup \{e_1, e_2\}$. We will modify T by diverting it into each $(P_1 \cup T')$-bridge group J of G such that J has vertices of attachment in P_1 to obtain the desired F_G-Tutte closed trail containing e_1, e_2 and e_3. Let J be a $(P_1 \cup T')$-bridge group of G such that J

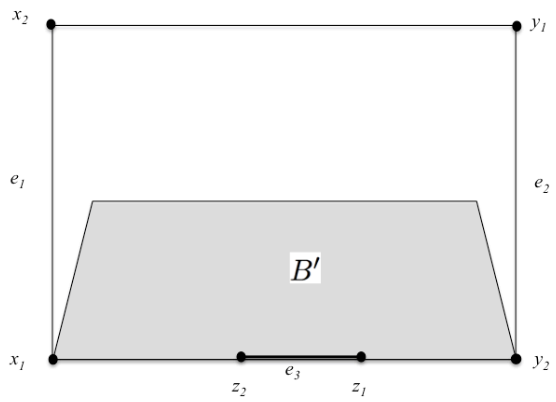


Figure 3.20: The structure of B'.

has vertices of attachment on P_1 and has more than three edges of attachment on $P_1 \cup T'$. By construction of B' and T', J has at most two vertices of attachment on T'. Then we will modify J as in Subcase 3.1, 3.2, 3.3 of Claim 6. Now, we have the desired F_G-Tutte closed trail containing e_1, e_2 and e_3. $\square$

The proof shows that G does not exist. Hence the theorem is true. ∎

We can extend Theorem 3.1(a) to connected graphs as follows.

Corollary 3.2 *Let G be a connected plane graph, F_G be the outer walk of G, $x, y \in V(G)$ and $z \in V(F_G)$. Then there exists an F_G-Tutte trail T in G from x to y such that either $z \in V(T)$ or the component of $G \backslash V(T)$ containing z has exactly one edge connecting it to T.*

Proof. Let S be a chain of edge-blocks from x to y which corresponds to a path in the edge-block tree of G. Then S is a sequence $x = a_1, B_1, b_1a_2, B_2, ..., b_{k-1}a_k, B_k, b_k = y$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$. We let $H = H_S$ and F_i be outer walks of B_i for all $1 \leq i \leq k$. If $z \notin V(H)$, then we let L be the component of $G \backslash H$ containing z. (See Fig. 3.21.) Note that L has only one edge connecting it to H. We let z' be the vertex in H that has a neighbor in L. (If $z \in V(H)$, we let $z = z'$.) Assume that $z \in V(B_m)$. By Theorem 3.1(a), for any $1 \leq i \leq k, i \neq m$, B_i has an F_i-Tutte trail T_i from a_i to b_i, and B_m has an F_m-Tutte trail T_m from a_m to b_m containing an edge of F_m incident to z'. Note that either $z \in V(T_m)$ or L has exactly one edge connecting it to T_m. Hence $T = (\bigcup_{i=1}^{k} B_i) \cup \{b_1a_2, b_2a_2, ..., b_{k-1}a_k\}$ is the desired F_G-Tutte trail in G. ∎

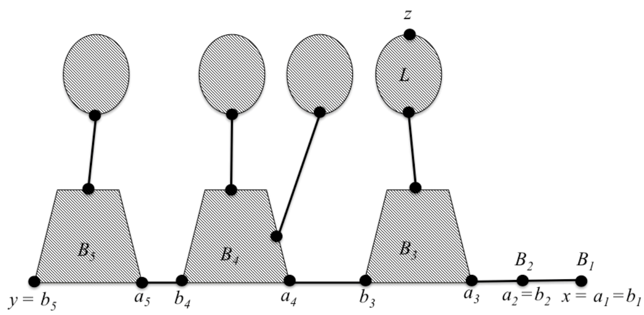


Figure 3.21: The structure of G in Corollary 3.2.

3.2 Some Tutte subgraphs of plane graphs

In this section, we show that a connected plane graph has a Tutte subgraph T such that T is the union of two or three disjoint trails. We then use this theorem to improve Theorem 3.1. First, we improve Corollary 3.2 to get the following lemma.

Lemma 3.3 *Let H be a connected plane graph, F_H be a facial walk of H, a, b, c be distinct vertices of F_H and $x \in V(F_H)\backslash\{a, c\}$. Suppose that for each end-edge-block K of H, we have $V(K) \cap \{a, b, c, x\} \neq \emptyset$. Let P be a subwalk of F_H from x to b with $a, c \notin V(P)$, and Q is a subwalk of F_H from a to b which contains P and $c \notin V(Q)$. Then there exists a P-Tutte subgraph T of H consisting of three edge-disjoint trails T_H, T_a and T_c such that T_H is a trail from x to b, T_a and T_c are closed trails containing a and c, respectively, $x \notin V(T_a) \cup V(T_c)$, and there is at most one component of $H\backslash V(T)$ which contains a vertex of F_H and has three edges connecting it to T.*

In addition, T is a Q-Tutte subgraph of H unless H has an edge-block K such that $H\backslash K$ has three components and $a, \{x, b\}, c$ belong to different components of $H\backslash K$.

Proof. By Corollary 3.2, there exists an F_H-Tutte trail T_H in H from x to b such that either $a \in V(T_H)$ or the component L_a of $H\backslash V(T_H)$ containing a has exactly one edge connecting it to T_H. If $a \in V(T_H)$, then we let $T_a = \{a\}$. Otherwise, let B_a be the edge-block of L_a containing a and a_1 be the vertex of B_a which is closest to T_H in H. If $c \in V(B_a)$, then we let $c = c^*$. Otherwise, let $c^* \in V(B_a)$ such that c^* has a neighbor in the component of $H\backslash B_a$ containing c. (Possibly, $a_1 = c^*$.) By Theorem 3.1(b, c), B_a has an F_{B_a}-Tutte closed trail T_a containing a, a_1 and c^*. Note that $T_H \cup T_a$ is an F_H-Tutte subgraph of H.

Let $T' = T_H \cup T_a$. If $c \in V(T')$, then we let $T_c = \{c\}$. Then $T = T' \cup \{c\}$ is the desired P-Tutte subgraph of H. Note that in the case, T is an F_H-Tutte subgraph of H. In particular, T is a Q-Tutte subgraph of H.

It remains to construct T_c when $c \notin V(T')$. Let L_c be the component of $H\backslash V(T')$ containing c. Then L_c has at most two edges connecting it to T'. Let B_c be the edge-block of L_c containing c. Since each end-edge-block of L_c contains c or a vertex which is adjacent to T, there are at most two vertices c_1 and c_2 of B_c which have neighbors in $H\backslash B_c$. (Possibly, $c_1 = c_2$.) By Theorem 3.1(b, c), B_c has an F_{B_c}-Tutte closed trail T_c containing c, c_1 and c_2. Note that if $x \notin B_a \cup B_c$, then $x \notin V(T_a) \cup V(T_c)$. Then $T = T_H \cup T_a \cup T_c$ is the desired P-Tutte subgraph of H.

It remains to show that there is at most one component of $H\backslash V(T)$ which contains a vertex of F_H and has three edges connecting it to T, and that T is a Q-Tutte subgraph of H. If $L_c = B_c$, then T is an F_H-Tutte subgraph of H. Hence we may assume that $L_c \neq B_c$. Then there is at most one component D of $L_c\backslash T_c$ such that D has three edges connecting it to T. Note that D contains a vertex of F_H if D exists. Next, we will consider T and D in two cases as follows.

Case 1: $L_c \cap L_a = \emptyset$. (See Fig. 3.22.)
Note that L_c has at most two edges connecting it to T_H.

Suppose B_c is an end-edge-block of L_c with no edge to T'. Since $L_c \neq B_c$, then D may exist and, if it does, then D has two edges connecting to T_H and one edge to T_c. (See Fig. 3.22(a).) Since Q does not contain a vertex of D, T is a Q-Tutte subgraph of H.

Suppose B_c is an end-edge-block of L_c with an edge connecting it to T', or B_c is not an end-edge-block of L_c. Then $L_c\backslash B_c$ has at most two components D_1, D_2 such that D_1 and/or D_2 have one edge connecting it to each T_H and T_c. (See Fig. 3.22(b).) Then T is an F_H-Tutte subgraph of H. In particular, T is a Q-Tutte subgraph of H.

Case 2: $L_c \cap L_a \neq \emptyset$. (See Fig. 3.23.)
Note that $L_c \subset L_a$, B_a and B_c are disjoint edge-blocks of L_a containing a and c, respectively. We will consider three subcases as follows.

Subcase 2.1: B_a is an end-edge-block of L_a but B_c is not an end-edge-block of L_a.
Note that L_c has one edge connecting it to each of T_H and T_a. Then $L_c\backslash B_c$ has at most two components D_1, D_2 such that D_1 and/or D_2 have two edges connecting them to T. Then T is an F_H-Tutte subgraph of H. In particular, T is a Q-Tutte subgraph of H.

Subcase 2.2: B_c is an end-edge-block of L_a but B_a is not an end-edge-block of L_a. (See Fig. 3.24.)
Note that L_c has one edge connecting it to T_a. Since $L_c \neq B_c$, the component $D_1 = L_c\backslash B_c$ has one edge connecting it to each of T_a and T_c. Then T is an F_H-

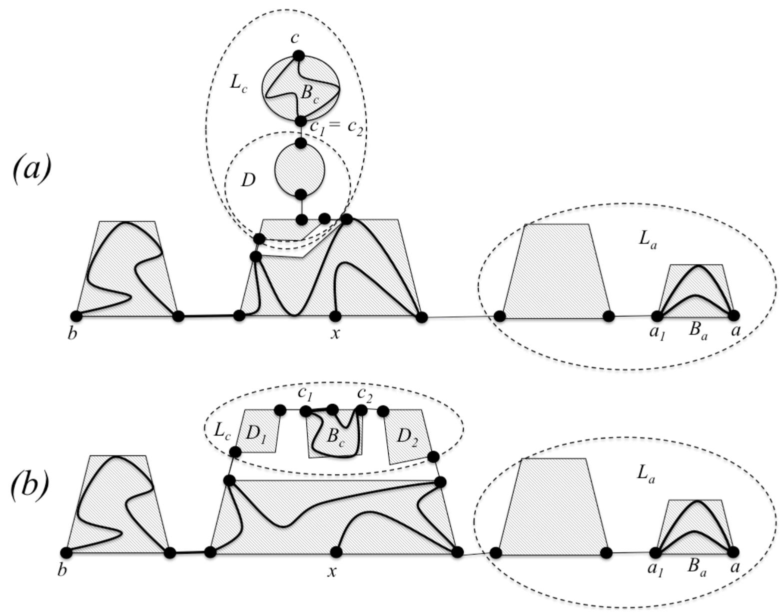


Figure 3.22: The structure of H, T_H, T_a and T_c in Case 1.

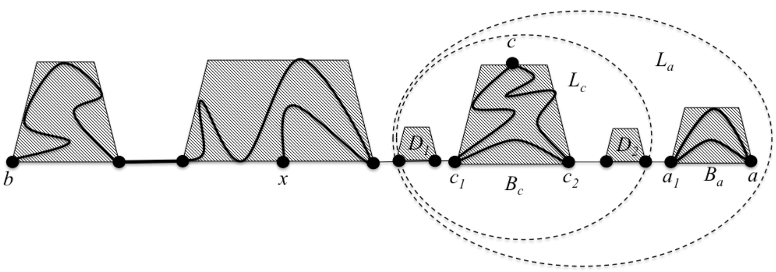


Figure 3.23: The structure of H, T_H, T_a and T_c in Subcase 2.1.

Tutte subgraph of H. In particular, T is a Q-Tutte subgraph of H.

Subcase 2.3: Both B_a and B_c are end-edge-blocks of L_a. (See Fig. 3.25.) Note that $B_a \neq B_c$, and L_c has one edge connecting it to each of T_H and T_a. Since $L_c \neq B_c$, D exists and D has one edge connecting to each of T_H, T_a and T_c. In this case, there is an edge-block K in D such that $H \backslash K$ has three components and $a, \{x, b\}, c$ belong to different components of $H \backslash K$.

■

Next, we introduce the following definition and prove Lemma 3.5 and Lemma

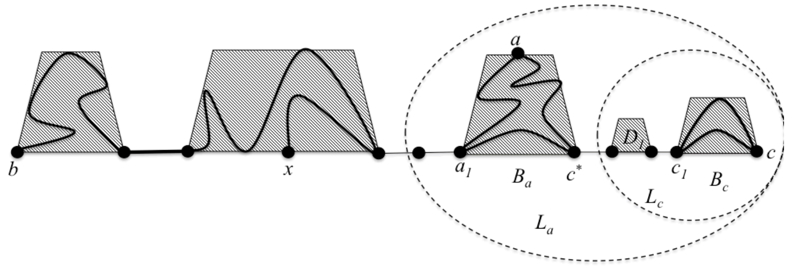


Figure 3.24: The structure of H, T_H, T_a and T_c in Subcase 2.2.

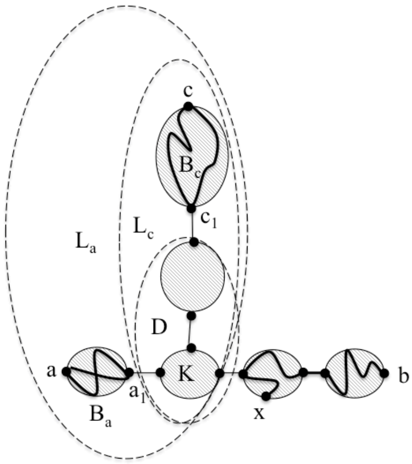


Figure 3.25: The structure of H, T_H, T_a and T_c in Subcase 2.3.

3.6. Lemma 3.5 and Lemma 3.6 will be used finding a Tutte trail in a projective plane graph.

Definition 3.4 *Let G be a graph and $u_1, u_2, v_1, v_2 \in V(G)$ with $u_1 \neq u_2$ and $v_1 \neq v_2$. We say that two edge-disjoint trails T_1 and T_2* connect $\{u_1, u_2\}$ and $\{v_1, v_2\}$ *in G if T_1 is a u_iv_j-trail and T_2 is a $u_{3-i}v_{3-j}$-trail for some $i, j \in \{1, 2\}$.*

Recall from Chapter 2, let C be a cycle in a plane graph G and $x, y \in V(C)$. We define $C[x, y]$ as a path of C from x to y in clockwise direction.

Lemma 3.5 *Let G be 2-connected plane graph, C be the outer cycle of G, $x, u \in V(C)$, $v \in V(C)\backslash\{u\}$, and $y \in V(G)\backslash\{x\}$. Then the following statements hold.*

(C1) Suppose $z \in \{u, v, y\} \cap V(C)$ and $D \in \{C[x, z], C[z, x]\}$. Then there exists a D-Tutte subgraph in G consisting of two edge-disjoint trails T_1 and T_2 such that T_1 and T_2 connect $\{x, y\}$ and $\{u, v\}$.

(C2) Suppose $w \in V(C)$ with $wx \in E(C)$. Then there exist:

(i) a C-flap H in G with attachments a, b, c such that $y, u, v \notin V(H)\backslash\{a, b, c\}$ and $x \in (V(H)\backslash\{a\}) \cup \{b\}$, and if H is non-trivial, then $w \in V(H)\backslash\{b\}$ and a, w, x, b appear in $C \cap H$ in this order;

(ii) a $(C\backslash(H\backslash\{a, b, c\}))$-Tutte subgraph in $G\backslash(H\backslash\{a, b, c\})$ consisting of two edge-disjoint trails T_1 and T_2 such that $a, c \in V(T_1 \cup T_2)$, and T_1 and T_2 connect $\{b, y\}$ and $\{u, v\}$. (See Fig. 3.26.)

Note that if H is trivial in (C2), then $V(H) = \{x\}$ and (C2) extends (C1) since it gives a C-Tutte subgraph consisting of two edge-disjoint trails connecting $\{x, y\}$ and $\{u, v\}$.

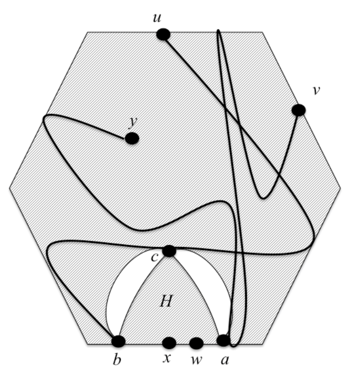


Figure 3.26: The structure of H, T_1 and T_2 in Lemma 3.5(C2).

Proof. Suppose $|\{x, y\} \cap \{u, v\}| = 1$. Then we can assume without loss of generality that $x = u$. Then, by Theorem 3.1(a), G has a C-Tutte trail T from y to v containing x. Then we can let $T = T_1 \cup T_2$ be such that T_1 and T_2 are edge-disjoint, T_1 is an xv-trail, and T_2 is an xy-trail. Thus (C1) holds and (C2) also holds when we let $H = \{x\}$.

Suppose $|\{x, y\} \cap \{u, v\}| = 2$, then, by Theorem 3.1(b, c), G has a C-Tutte closed trail T' containing x and y. Then $T' = T_1' \cup T_2'$ where T_1' and T_2' are edge-disjoint trails from x to y. Thus (C1) holds and (C2) also holds when we let $H = \{x\}$. Hence we may assume that x, y, u, v are all distinct.

We prove the theorem by using induction on $|V(G)|$. If $|V(G)| \leq 3$, then $\{x, y\} \cap \{u, v\} \neq \emptyset$ and the lemma holds. So we assume $|V(G)| \geq 4$ and proceed to the induction step.

We prove $(C1)$ and $(C2)$, simultaneously. The proof has two parts as follows:

(I) $(C1)$ is true when $|V(G)| = n$ if $(C2)$ is true when $|V(G)| \leq n$.

(II) $(C2)$ is true when $|V(G)| = n$ if $(C1)$ and $(C2)$ are true when $|V(G)| < n$.

Proof of (I). Choose $w \in V(C)$ such that $wx \in E(C\backslash D)$. By induction on (C2), there exist a C-flap H in G with attachments a, b, c such that $u, v, y \notin V(H)\backslash\{a, b, c\}$ and $x \in (V(H)\backslash\{a\})\cup\{b\}$, and a $(C\backslash(H\backslash\{a, b, c\}))$-Tutte subgraph in $G\backslash(H\backslash\{a, b, c\})$ consisting of two edge-disjoint trails T_1 and T_2 such that $a, c \in V(T_1 \cup T_2)$, T_1 and T_2 connect $\{b, y\}$, and $\{u, v\}$. Furthermore, if H is non-trivial, then $w \in V(H)\backslash\{b\}$ and a, w, x, b appear in $C \cap H$ in this order. Note that $b \in V(D)$.

If H is trivial with $x = a = b = c$, then $T_1 \cup T_2$ is the desired D-Tutte subgraph in G. Hence we may assume that H is non-trivial. Let P be the path of D from x to b. By Lemma 3.3, there exists a P-Tutte subgraph T of H consisting of three edge-disjoint trails T_H, T_a and T_c such that T_H is a trail from x to b, T_a and T_c are closed trails containing a and c, respectively. Then $K = T_1 \cup T_2 \cup T$ is a D-Tutte subgraph of G and K can be decomposed into two edge-disjoint trails T_1' and T_2' such that T_1' and T_2' connect $\{x, y\}$ and $\{u, v\}$. Then $T_1' \cup T_2'$ is the desired D-Tutte subgraph in G. (See Fig. 3.27.) ■

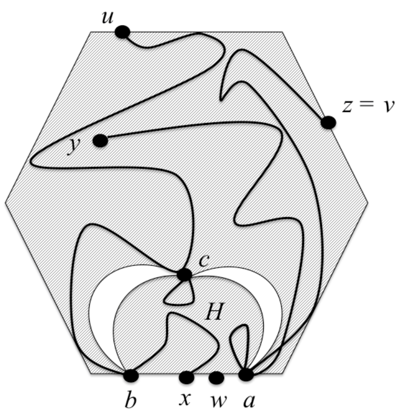


Figure 3.27: The structure of T_1' and T_2' in Lemma 3.5(C1).

Proof of (II). Suppose that x, u, v appear in clockwise order.

Case 1: G has a 2-separation (G_1, G_2) with $x \in V(G_1) \cap V(G_2)$.

Suppose that $V(G_1) \cap V(G_2) = \{x, z\}$, $E(G_1) \cap E(C) \neq \emptyset$, and $u \in V(G_1)$. Note that if $z \notin V(C)$, then $E(C) \cap E(G_2) = \emptyset$ and G_1 is 2-connected. (See Fig. 3.28.)

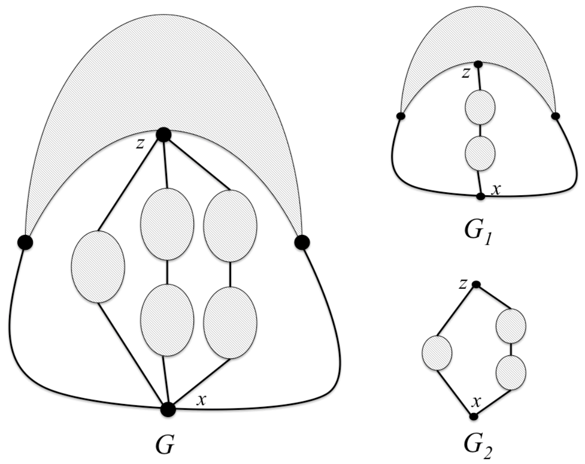


Figure 3.28: The structure of G, G_1 and G_2 in Case 1 when $z \notin V(C)$. Note that the 2-separation (G_1, G_2) is not unique.

Let $G_1 = G_R$ where R is a chain of blocks $x = c_0, A_1, c_1, A_2, ..., c_{n-1}, A_n, c_n = z$. (See Fig. 3.29.) By Theorem 3.1(a), A_i has an F_{A_i}-Tutte trail T_{A_i} from c_{i-1} to c_i for all $1 \leq i \leq n$.

Let $G_2 = H_S$ where S is a chain of edge-blocks $x = a_1, B_1, b_1 a_2, B_2, ..., b_{k-1} a_k, B_k, b_k = z$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$. (See Fig. 3.29.) By Theorem 3.1(a), B_i has an F_{B_i}-Tutte trail T_{B_i} from a_i to b_i for each $1 \leq i \leq k$.

Case 1.1: $u, v \in V(G_1) \backslash \{z\}$ and $y \in V(G_1)$.

In this case, either $E(C) \cap E(G_2) \neq \emptyset$, or $E(C) \cap E(G_2) = \emptyset$. Let G_1' be a new graph obtained from G_1 by adding a new edge e_1 from x to z such that $E(F_{G_1} \cap C) \subseteq E(F_{G_1'})$. Let $w_1 = w$ if $w \in V(G_1)$; otherwise $w_1 = z$. Note that $w_1 x \in E(F_{G_1'})$. By induction on (C2), there exist: an $F_{G_1'}$-flap H_1 in G_1' with attachments a, b, c such that $y, u, v \notin V(H_1) \backslash \{a, b, c\}$ and $x \in (V(H_1) \backslash \{a\}) \cup \{b\}$, and if H_1 is non-trivial, then $w_1 \in V(H_1) \backslash \{b_1\}$ and a_1, w_1, x, b_1 appear in $F_{G_1'} \cap H_1$ in this order; an $(F_{G_1'} \backslash (H_1 \backslash \{a, b, c\}))$-Tutte subgraph in $G_1' \backslash (H_1 \backslash \{a, b, c\})$ consisting of two edge-

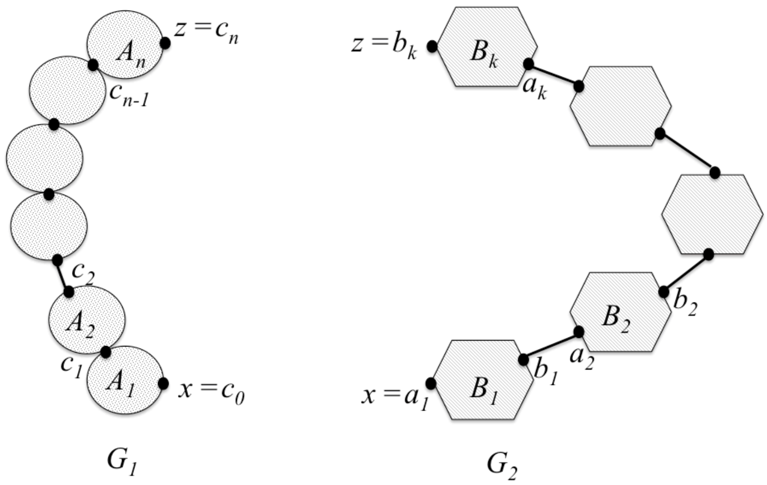


Figure 3.29: The structure of G_1 and G_2 in Case 1.

disjoint trails T_1' and T_2' such that $a, c \in V(T_1' \cup T_2')$, and T_1' and T_2' connect $\{b, y\}$ and $\{u, v\}$.

Subcase 1.1.1: $e_1 \notin E(T_1' \cup T_2')$.
First, suppose that $e_1 \in E(H_1)$. (See Fig. 3.30.) Then $H = (H_1 - e_1) \cup G_2$ is the desired C-flap in G and $T_1' \cup T_2'$ is also the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte subgraph of $G\backslash(H\backslash\{a, b, c\})$.

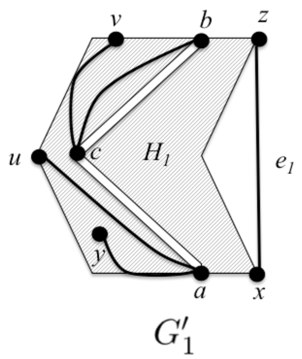


Figure 3.30: The structure of G_1 and G_2 in Subcase 1.1.1 when $e_1 \in V(H_1)$.

Next, suppose $e_1 \notin V(H_1)$. Then $x = b$. Note that H_1 is either trivial, or non-trivial with $w = w_1$. Assume without loss of generality that T_1' is a trail from x to u. By Theorem 3.1(a), B_1 has an F_{B_1}-Tutte trail T_x from x to x containing

b_1, and B_k has an F_{B_k}-Tutte trail T_z from z to z containing a_k.(If $B_1 = B_k$, we let $T_z = \{z\}$.) Suppose that $z \in V(T_1' \cup T_2')$. If $z \in V(T_1')$, we let $T_1 = T_1' \cup T_x \cup T_z$ and $T_2 = T_2'$; otherwise, we let $T_1 = T_1' \cup T_x$ and $T_2 = T_2' \cup T_z$. Then H_1 is the desired C-flap of G and $T_1 \cup T_2$ is the desired $(C\backslash(H_1\backslash\{a,b,c\}))$-Tutte subgraph of $G\backslash(H_1\backslash\{a,b,c\})$.(See Fig. 3.31.)

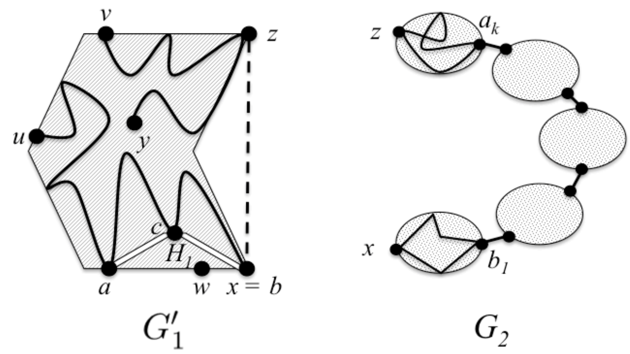


Figure 3.31: The structure of G_1' in Subcase 1.1.1 when $e_1 \notin V(H_1)$ and $z \in V(T_1' \cup T_2')$.

Hence suppose that $z \notin V(T_1' \cup T_2')$. If $z \in V(C)$ (respectively, $z \notin V(C)$), the component B of $G_1'\backslash V(T_1' \cup T_2')$ containing z has two edges including the edge zx (respectively, at most three edges including the edge zx) connecting it to $V(T_1' \cup T_2')$. Let $K = G_2 \cup B$ and L_x be the edge-block of K containing x. (Possibly, $L_x = B_1$.)

If $z \in V(C)$, then we let $v_x \in L_x$ such that $v_x \neq x$ and v_x has a neighbor in $G\backslash L_x$. (Possibly, $v_x = b_1$.) By Theorem 3.1(a), L_x has an F_{L_x}-Tutte trail T_x' from x to x containing v_x. (See Fig. 3.32.) Note that if $K\backslash L_x \neq \emptyset$, $K\backslash L_x$ has at most two edges connecting it to $T_1' \cup T_2' \cup T_x'$. Then we let $T_1'' = T_1' \cup T_x'$.

If $z \notin V(C)$, then we let $v_1, v_2 \in L_x$ such that $v_1 \neq x, v_2 \neq x$, and both vertices have a neighbor in $G\backslash L_x$. (Possibly, $v_1 = v_2$.) By Theorem 3.1(a), L_x has an F_{L_x}-Tutte trail T_x'' from x to x containing v_1. (See Fig. 3.33.) Note that if $v_2 \notin V(T_x'')$, the component $G\backslash V(T_1' \cup T_2' \cup T_x'')$ containing v_2 has at most three edges connecting it to $V(T_1' \cup T_2' \cup T_x'')$. Then we let $T_1'' = T_1' \cup T_x''$.

Hence H_1 is the desired C-flap of G and $T_1'' \cup T_2'$ is the desired $(C\backslash(H_1\backslash\{a,b,c\}))$-Tutte subgraph of $G\backslash(H_1\backslash\{a,b,c\})$. (See Fig. 3.32, 3.33.)

Subcase 1.1.2: $e_1 \in E(T_1' \cup T_2')$.

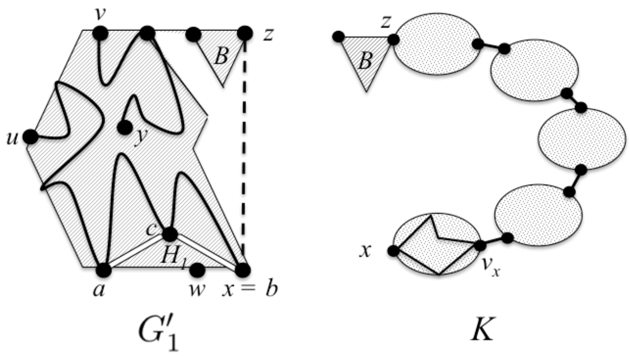


Figure 3.32: The structure of G_1' and K in Subcase 1.1.1 when $e_1 \notin V(H_1)$ and $z \notin V(T_1' \cup T_2')$ and $z \in V(C)$.

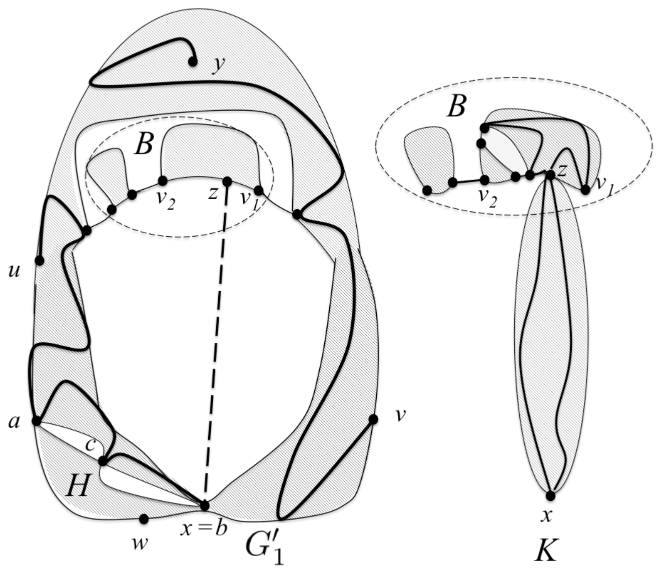


Figure 3.33: The structure of G_1' and K in Subcase 1.1.1 when $e_1 \notin V(H_1)$ and $z \notin V(T_1' \cup T_2')$ and $z \notin V(C)$.

Then $b = x$. Note that H_1 is either trivial, or non-trivial with $w = w_1$ Assume without loss of generality that $e_1 \in E(T_1')$. Let $T'' = \{b_1a_2, b_2a_3, ..., b_{k-1}a_k\} \cup \bigcup_{i=1}^{k} T_{B_i}$. Note that T'' is an F_{G_2}-Tutte trail from x to z in G_2. Let $T_1 = (T_1' - e_1) \cup T''$. Then H_1 is the desired C-flap of G and $(T_1 \cup T_2')$ is also the desired $(C \backslash (H_1 \backslash \{a, b, c\}))$-Tutte subgraph of $G \backslash (H_1 \backslash \{a, b, c\})$.(See Fig. 3.34.)

Case 1.2: $u, v \in V(G_1) \backslash \{z\}$ and $y \in V(G_2) \backslash \{z\}$.

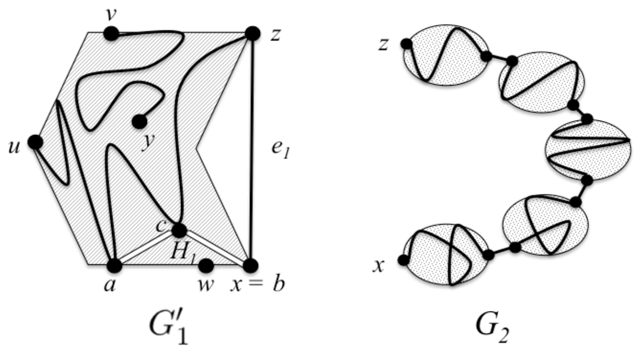


Figure 3.34: The structure of G_1' and G_2 in Subcase 1.1.2.

In this case, we can find T_1, T_2, H such that H is trivial. Suppose that $y \in V(B_m)$ for some $1 \leq m \leq k$. Note that if $z \notin V(C)$, then, by assumption, G_1 is 2-connected. By Theorem 3.1(a), B_m has an F_{B_m}-trail T'_{B_m} from y to b_m containing a_m, and if $m \neq 1$, B_1 has an F_{B_1}-Tutte trail T_x from x to x containing b_1. (We let $T_x = \{x\}$ if $m = 1$.) We let $T_{yz} = T'_{B_m} \cup \{b_m a_{m+1}, b_{m+1} a_{m+2}, ..., b_{k-1} a_k\} \cup \bigcup_{i=m+1}^{k} T_{B_i}$. Suppose that $u \in V(A_p)$ and $v \in V(A_q)$ for some $1 \leq p \leq q \leq n$.

Subcase 1.2.1: $p \neq q$.

By Theorem 3.1(a), A_p has an F_{A_p}-Tutte trail T' from c_{p-1} to u containing c_p, and A_q has a Tutte trail T'' from c_q to v containing c_{q-1}. Let $s = c_p$, $t = c_{q-1}$, and if $p < q-1$, let $D = \bigcup_{i=p+1}^{q-1} A_i$; otherwise, let $D = \{s\} = \{t\}$. Next, we will define T_s and T_t as follows.

- When $D = \{s\}$, we let $T_s = T_t = \{s\}$.
- When D is 2-edge-connected, we let $T_t = \{t\}$. By Theorem 3.1, we can choose T_s to be an F_D-Tutte trail of D from s to s containing t.
- When D is not 2-edge-connected, we let P_s (respectively, P_t) be an edge-block of D containing s(respectively, t). Let v_s(respectively, v_t) be the vertex of P_s(respectively, P_t) such that $v_s \neq s$(respectively, $v_t \neq t$) and v_s(respectively, v_t) has a neighbor in $D \backslash P_s$(respectively, $D \backslash P_t$). By Theorem 3.1, P_s has an F_{P_s}-Tutte trail T_s from s to s containing v_s, and P_t has an F_{P_t}-Tutte trail T_t from t to t containing v_t.

In each case, we let $T_1 = T_x \cup T_s \cup T' \cup \bigcup_{i=1}^{p-1} T_{A_i}$ and $T_2 = T_{yz} \cup T_t \cup T'' \cup \bigcup_{i=1}^{q-1} T_{A_i}$. Note that T_1 is an xu-trail and T_2 is a yv-trail. Hence $H = x$ is the desired C-flap of G and $T_1 \cup T_2$ is the desired C-Tutte subgraph of G.(See Fig. 3.35.)

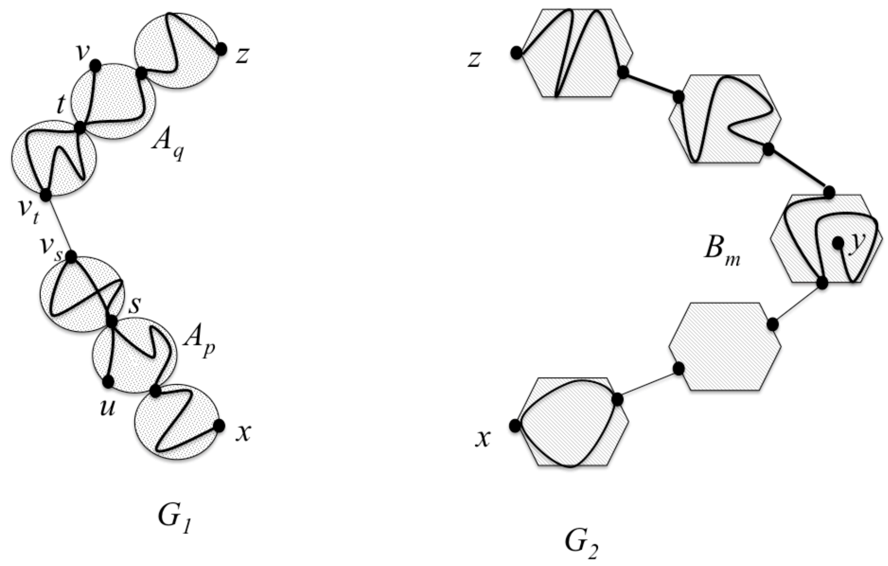


Figure 3.35: The structure of G_1 and G_2 in Subcase 1.2.1.

Subcase 1.2.2: $p = q$.

Note that $C[c_{p-1}, c_p] = C \cap B_p$ and c_{p-1}, u, v, c_p appear in $C[c_{p-1}, c_p]$ in this clockwise order. By induction on (C1), B_p has a $C[c_{p-1}, c_p]$-Tutte subgraph consisting of two edge-disjoint trails T' and T'' such that T' and T'' connect $\{c_{p-1}, c_p\}$ and $\{u, v\}$. Assume without loss of generality that T' is a $c_{p-1}u$-trail. We let $T_1 = T_x \cup T' \cup \bigcup_{i=1}^{p-1} T_{A_i}$ and $T_2 = T_{yz} \cup T'' \cup \bigcup_{i=p+1}^{n} T_{A_i}$. Hence $H = x$ is the desired C-flap of G and $T_1 \cup T_2$ is the desired C-Tutte subgraph of G. (See Fig. 3.36.)

Case 1.3: $u \in V(G_1)$ and $v \in V(G_2)$.

Note that $E(C) \cap E(G_2) \neq \emptyset$. We again construct T_1, T_2, H such that H is trivial. By symmetry, we can assume that $y \in V(G_1)$. Suppose that $y \in V(A_p)$, $u \in V(A_q)$ for some $p, q \in \{1, 2, ..., n\}$, and $v \in V(B_m)$ for some $m \in \{1, 2, ..., k\}$.

Subcase 1.3.1: $p \neq q$.

Assume without loss of generality that $p \leq q$. By Theorem 3.1(a), A_p has an F_{A_p}-Tutte trail T'_{A_p} from y to c_p containing c_{p-1}, A_q has an F_{A_q}-Tutte trail T'_{A_q} from u

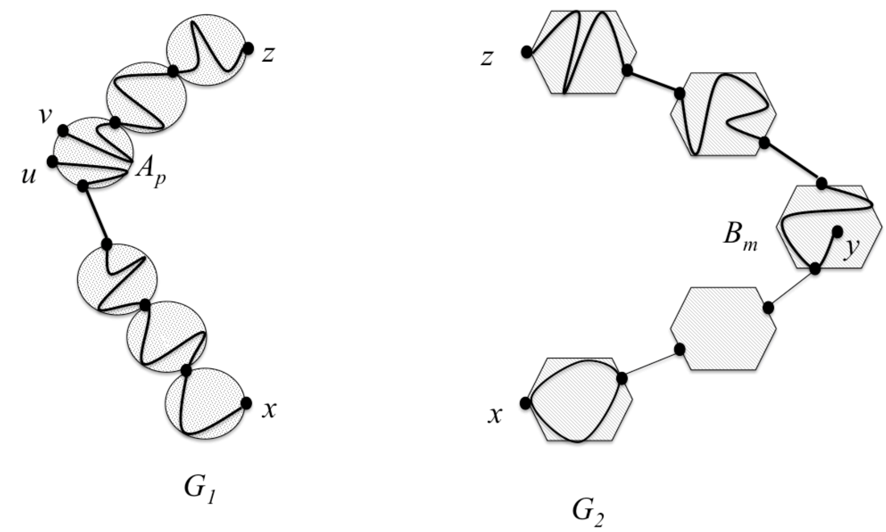


Figure 3.36: The structure of G_1 and G_2 in Subcase 1.2.2.

to c_{q-1} containing c_q, and B_m has an F_{B_m}-Tutte trail T'_{B_m} from v to a_m containing b_m. Let $c_{p-1} = r$ and $c_q = s$. Let $D_1 = \bigcup_{i=1}^{p-1} A_i$. Then $x, r \in V(D_1)$. We use the same method as in Case 1.2.1 to define T_x and T_r in D_1. If $m \neq k$, we also let $D_2 = B_k \cup \bigcup_{i=q+1}^{n} A_i$; otherwise, let $D_2 = \bigcup_{i=q+1}^{n} A_i$. We also use the same method in Subcase 1.2.1 to define T_s in D_2. Let $T_1 = T_x \cup T'_{B_m} \cup \bigcup_{i=1}^{m-1} T_{B_i}$ and $T_2 = T_r \cup T_s \cup T'_{A_p} \cup T'_{A_q} \cup \bigcup_{i=p+1}^{q-1} T_{A_i}$. Hence $H = \{x\}$ is the desired C-flap of G and $T_1 \cup T_2$ is the desired C-Tutte subgraph of G.(See Fig. 3.37.)

Subcase 1.3.2: $p = q$.

In this case, we use the similar proof as in Subcase 1.2.2. By Theorem 3.1(a), B_m has an F_{B_m}-trail T'_{B_m} from v to b_m containing a_m, and if $m \neq 1$, B_1 has an F_{B_1}-Tutte trail T_x from x to x containing b_1. (We let $T_x = \{x\}$ if $m = 1$.) We let $T_{vz} = T'_{B_m} \cup \bigcup_{i=m+1}^{k} T_{B_i}$.

Since c_{p-1}, u, c_p appear in $C[c_{p-1}, c_p]$ in this clockwise order. By induction on (C1), B_p has a $C[c_{p-1}, c_p]$-Tutte subgraph consisting of two edge-disjoint trails T' and T'' such that T' and T'' connect $\{c_{p-1}, y\}$ and $\{u, c_p\}$. Assume without loss of generality that c_{p-1} is an end vertex of T'. We let $T^* = T_x \cup \bigcup_{i=1}^{p-1} T_{A_i}$ and $T^{**} = T_{vz} \cup \bigcup_{i=p+1}^{n} T_{A_i}$. If u is an end vertex of T', we let $T_1 = T^* \cup T'$ and $T_2 = T^{**} \cup T''$; otherwise, we let $T_1 = T^{**} \cup T'$ and $T_2 = T^* \cup T''$. Hence $H = x$ is

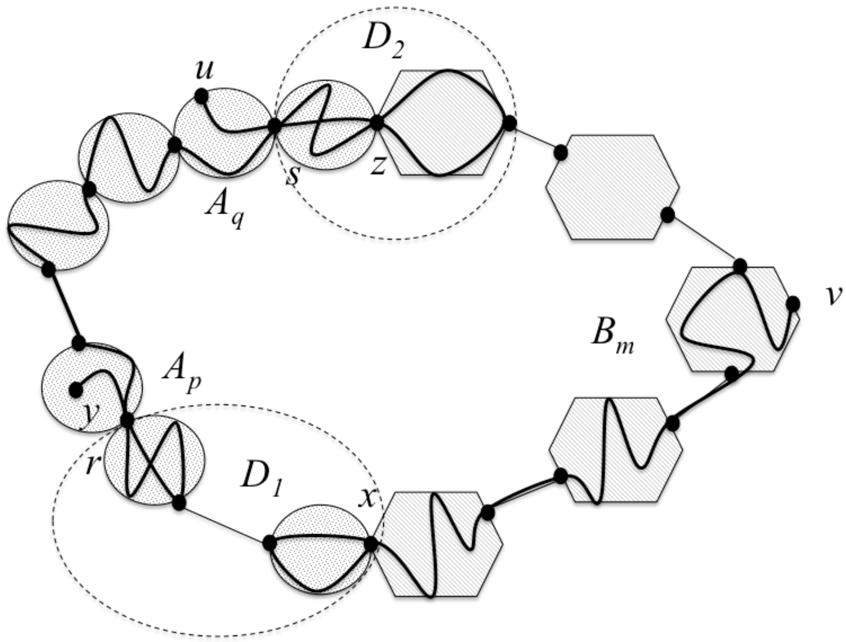


Figure 3.37: The structure of G_1 and G_2 in Subcase 1.3.1.

the desired C-flap of G and $T_1 \cup T_2$ is the desired C-Tutte subgraph of G.(See Fig. 3.38 and 3.39.)

Case 2: $G \backslash \{x\}$ is 2-connected.

Let $G^* = G\backslash\{x\}$. Let z be the neighbor of x in C with $z \neq w$. Let C^* be outer walk of G^*. We let $w^* \in V(C^*)$ with $w^*z \in E(C^*\backslash C)$. By induction on (C2), there exist: a C^*-flap H^* in G^* with attachments a^*, b^*, c^* such that $u, v, y \notin V(H^*)\backslash\{a^*, b^*, c^*\}$ and $z \in (V(H^*)\backslash\{a^*\}) \cup \{b^*\}$, and if H^* is non-trivial, then $w^* \in V(H^*)\backslash\{b^*\}$ and a^*, w^*, x, b^* appear in $C^* \cap H^*$ in this order; a $(C^*\backslash(H^*\backslash\{a^*, b^*, c^*\})$-Tutte subgraph in $(G^*\backslash(H^*\backslash\{a^*, b^*, c^*\}$ consisting of two edge-disjoint trails T_1^* and T_2^* such that $a^*, c^* \in V(T_1^* \cup T_2^*)$, and T_1^* and T_2^* connect $\{b^*, y\}$ and $\{u, v\}$.

Case 2.1: H^* is trivial.

Then $z = a^* = b^* = c^*$. Assume without loss of generality that z is an end vertex of T_1^*. If $w \notin V(T_1^* \cup T_2^*)$, we let K be the component of $G^*\backslash V(T_1^* \cup T_2^*)$ containing w.

Case 2.1.1: $w \notin V(T_1^* \cup T_2^*)$ and $V(C^*\backslash C) \subset V(K)$.

Note that z is a neighbor of K in $V(T_1^* \cup T_2^*)$. Let K' be the subgraph of G

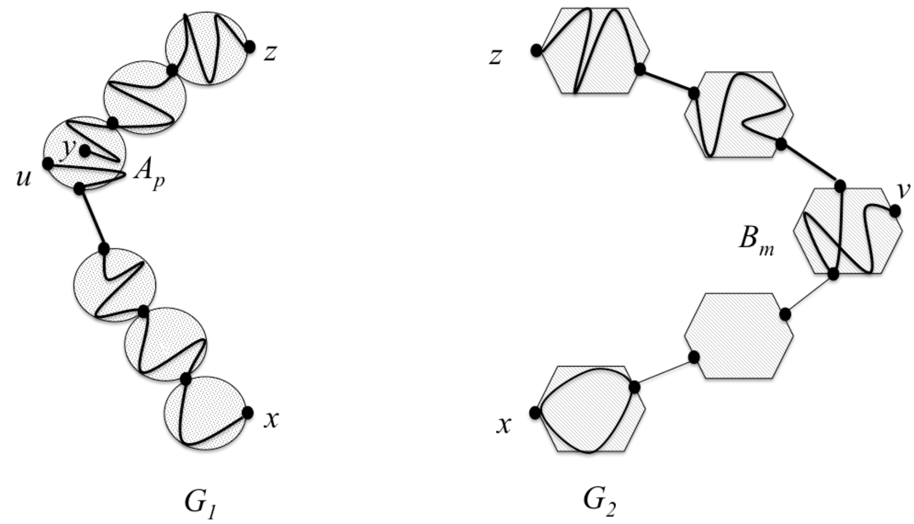


Figure 3.38: The structure of G_1 and G_2 in Subcase 1.3.2 when T_1 is a trail from x to u.

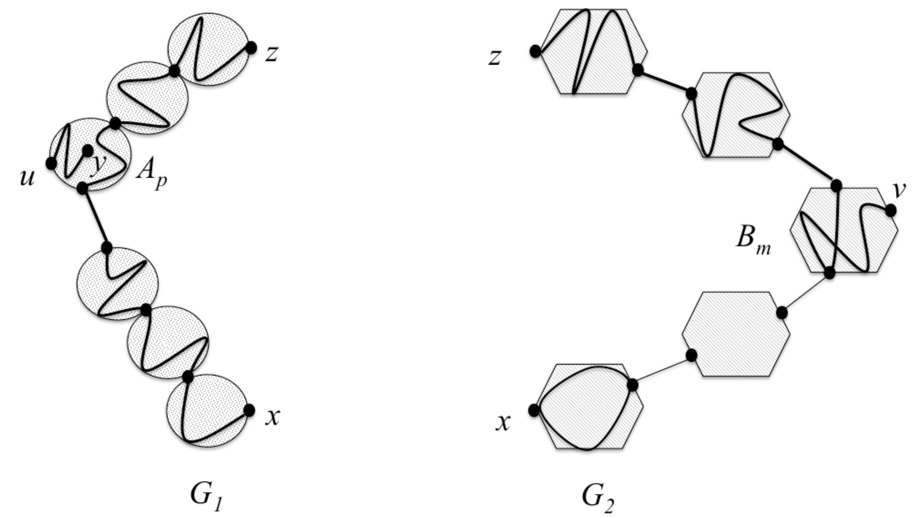


Figure 3.39: The structure of G_1 and G_2 in Subcase 1.3.2 when T_1 is a trail from x to v.

induced by $V(K) \cup \{x, z\}$ and $B_{K'}$ be the edge-block of K' containing w. Note that $x, z \in V(B_{K'})$. Let $d \in V(B_{K'})\backslash\{z\}$ such that d has neighbor in $G\backslash B_{K'}$. By Theorem 3.1(a), $B_{K'}$ has an $F_{B_{K'}}$-Tutte trail $T_{K'}$ from x to z containing d. (See Fig. 3.40.) Note that the component of $K'\backslash B_{K'}$ has two edges connecting it to

$T_1^* \cup T_2^* \cup T_{K'}$. Then we let $T_1 = T_1^* \cup T_{K'}$. Then $H = \{x\}$ is the desired C-flap of G and $T_1 \cup T_2^*$ is the desired C-Tutte subgraph of G.

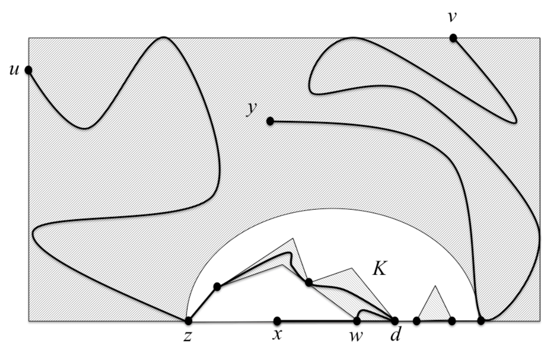


Figure 3.40: The structure of T_1^*, T_2^* and $T_{K'}$ in G in Case 2.1.1.

Case 2.1.2: $w \in V(T_1^* \cup T_2^*)$, or $V(C^* \backslash C) \not\subset V(K)$.
In this case, we will show that the desired C-flap H is either trivial, or non-trivial with attachments a, x, c for some vertices $a, c \in V(G)$. Let $J_1, J_2, ..., J_n$ be the distinct components of $G^* \backslash V(T_1^* \cup T_2^*)$ such that $J_i \neq K$ and J_i contains a vertex of $C^* \backslash C$ for all $1 \leq i \leq n$. Then J_i has exactly two edges connecting it to $T_1^* \cup T_2^*$. If J_i has at most one edge connecting it to x for some $1 \leq i \leq n$, we let $T_{J_i} = \emptyset$.

Next, suppose that J_i has at least two edges connecting it to x for some $1 \leq i \leq n$. Let J_i' be the subgraph of G induced by $V(J_i) \cup \{x\}$ and L be the edge-block of J_i containing x. Then $J_i' \backslash L$ has two components L_1', L_2'. (Possibly $L_1' = L_2'$.) Let $x_i \in V(L)$, for $i = 1, 2$ such that x_i has a neighbor in L_i'. (If $L_1' = L_2'$, then $x_1 = x_2$.) By Theorem 3.1(a), L has an F_L-Tutte trail T_{J_i} from x to x containing x_1. (See Fig. 3.41.) Note that if $x_2 \notin V(T_{J_i})$, the components Q of $L \backslash V(T_{J_i})$ containing x_2 has at most two edges connecting it to T_{J_i} and then $L_2' \cup Q$ has at most three edges connecting it $T_1^* \cup T_2^* \cup T_{J_i}$.

If $w \in V(T_1^* \cup T_2^*)$, we let $H = \{x\}$, and if $w \notin V(T_1^* \cup T_2^*)$, we let H be the $(T_1^* \cup T_2^* \cup \{x\})$-bridge of G containing w. Note that if $w \notin V(T_1^* \cup T_2^*)$, then $K \subset H$ and H is the C-flap with three vertices of attachment and one of which is x. Let a and c be the other two vertices of attachment of H such that $a \in V(C)$ and $c \in V(C^* \backslash C)$.

In both cases, we let $T_1 = T_1^* \cup \{xz\} \cup \bigcup_{i=1}^{n} T_{J_i}$. (See Fig. 3.42.) Then H is the desired C-flap of G with attachments a, x, c and $(T_1 \cup T_2^*)$ is also the desired

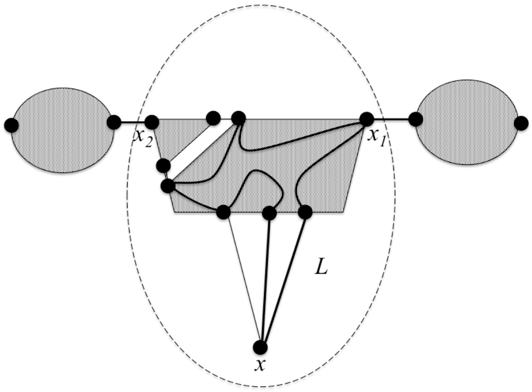


Figure 3.41: The structure of T_{J_i} in J_i' in Case 2.1.2.

$(C\backslash(H\backslash\{a,x,c\}))$-Tutte subgraph of $G\backslash(H\backslash\{a,x,c\})$.

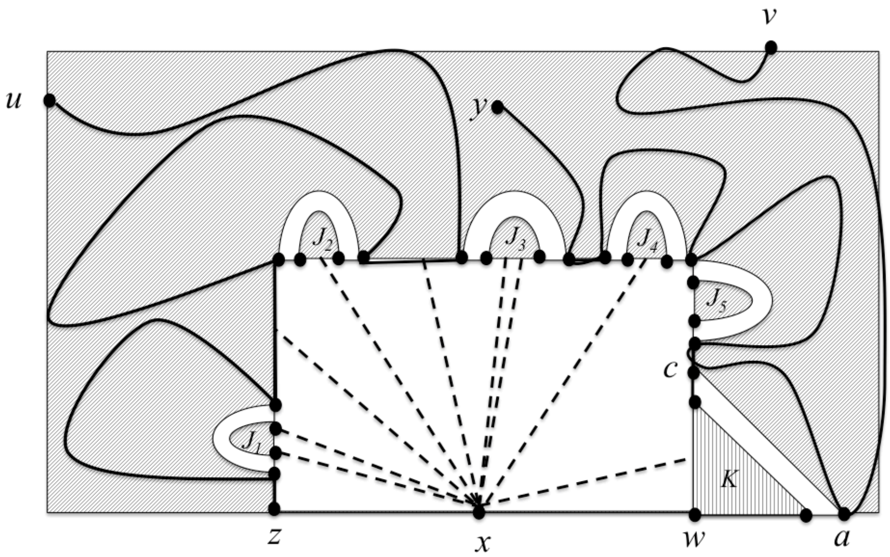


Figure 3.42: The structure of H, T_1 and T_2^* in G in Case 2.1.2 when $w \notin V(T_1 \cup T_2^*)$.

Case 2.2: H^* is non-trivial with $a^* \in V(C^*\backslash C)$ and $b^* \in V(C)$.
In this case, we will show that the desired C-flap H is either trivial, or non-trivial with attachments a, x, c for some $a, c \in V(G)$. Assume without loss of generality that b^* is the end vertices of T_1^*. Let $v_1, v_2, ..., v_k$ be vertices of $V(H^*)$ such that they are neighbors of x in G. We also assume that $z = v_1, v_2, ..., v_k$ appear in $C^* \cap H^*$ in

this order. (See Fig. 3.43.)

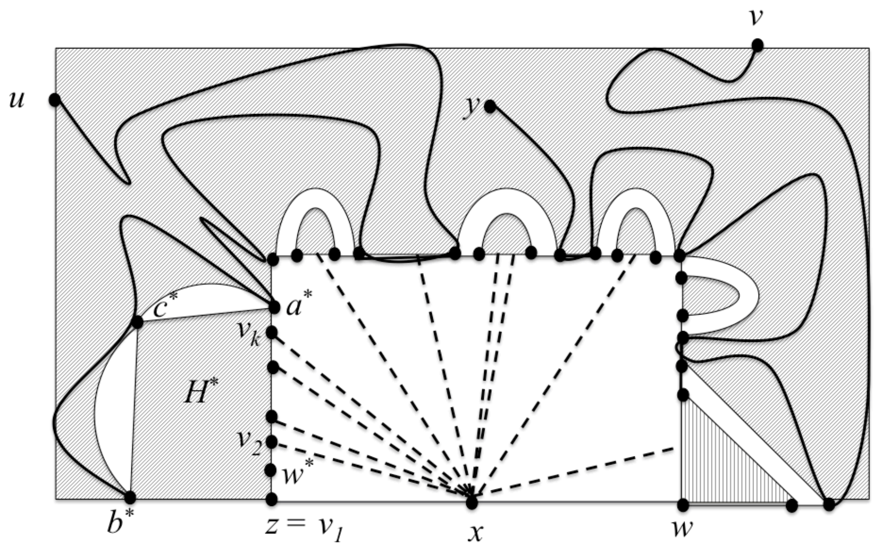


Figure 3.43: The structure of H^*, T_1^* and T_2^* in G in Case 2.2.

Let P be the path of $F_{H^*} \cap C^*$ from v_k to b^*. By Corollary 3.2, there exists an F_{H^*}-Tutte trail T_{H^*} in H^* from v_k to b^* such that either $z \in V(T)$ or the component M of $G \backslash V(T)$ containing z has at exactly one edge connecting it to T. Since $z \in V(P)$, M cannot have just one edge connecting it to T_{H^*}. This implies that $z \in V(T_{H^*})$.

Since a^* is a vertex of F_{H^*}, $a^* \in V(T_{H^*})$ or the component B of $G \backslash V(T)$ containing a^* has at most two edges connecting it to T_{H^*}. Let A be the edge-block of B containing a^*, and $x_1, x_2 \in V(A)$ be such that $x_1 \neq a^*$, $x_2 \neq a^*$ and both vertices have a neighbor in $H^* \backslash A$. (Possibly, $x_1 = x_2$.) By Theorem 3.1(a), A has an F_A-Tutte trail T_{a^*} from a^* to a^* containing x_1. If $x_2 \notin T_{a^*}$, the component of $A \backslash T_{a^*}$ containing x_2 has two edges connecting it to T_{a^*}, and then the component of $H^* \backslash V(T_{H^*} \cup T_{a^*})$ containing x_2 has three edges connecting it to $T_{H^*} \cup T_{a^*}$. Since c^* is a vertex of F_{H^*}, $c^* \in V(T_{H^*} \cup T_{a^*})$ or the component of $G \backslash V(T)$ containing c^* has at most two edges connecting it to $T_{H^*} \cup T_{a^*}$. Then we use exactly the same method again to get T_{c^*}. Hence $T = T_{H^*} \cup T_{a^*} \cup T_{c^*}$ is an P-Tutte subgraph of H^*. Then $T_1^* \cup T_2^* \cup T \cup \{v_k x\}$ can be decomposed into two edge-disjoint trails T_1 and T_2 such that x is the end vertex of T_1, and T_1 and T_2 connect $\{x, y\}$ and $\{u, v\}$. (See Fig. 3.44.)

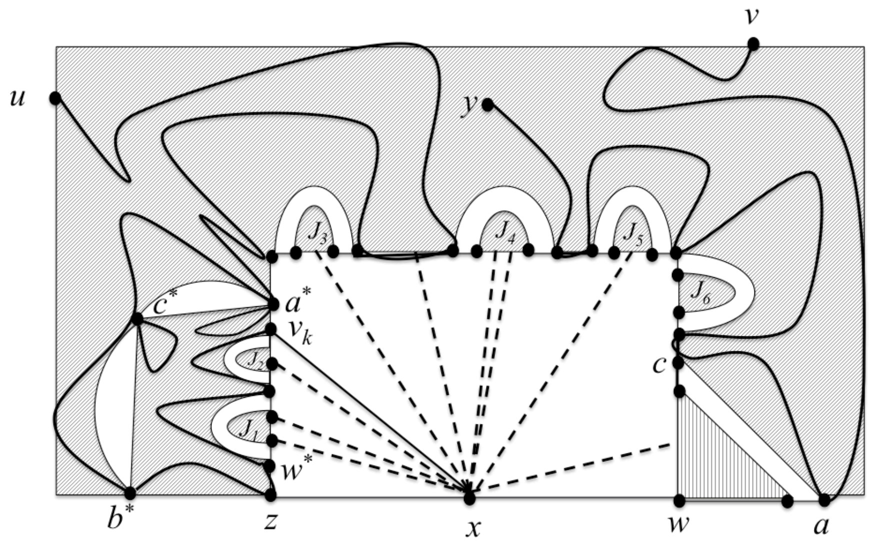


Figure 3.44: The structure of T_1 and T_2 in G in Case 2.2.

Let $J_1, J_2, ..., J_n$ be the distinct components of $G^*\backslash V(T_1 \cup T_2)$ such that $J_i \neq K$ and J_i contains a vertex of $C^*\backslash C$ for all $1 \leq i \leq n$. (Possibly, there is no such component.) Then we use the same method in Case 2.1.2 to define T_{J_i}.

If $w \in V(T_1 \cup T_2)$, we let $H = \{x\}$, and if $w \notin V(T_1 \cup T_2)$, we let H be the $(T_1 \cup T_2)$-bridge of G containing w. Note that H is the C-flap with three vertices of attachment and one of which is x. Let a and c be the other two vertices of attachment of H such that $a \in V(C)$ and $c \in V(C^*\backslash C)$.

In both cases, we let $T_1' = T_1 \cup \bigcup_{i=1}^{n} T_{J_i}$. (See Fig. 3.45.) Then H is the desired C-flap of G with attachments a, x, c and $(T_1' \cup T_2)$ is also the desired $(C\backslash(H\backslash\{a, x, c\}))$-Tutte subgraph of $G\backslash(H\backslash\{a, x, c\})$.

Case 2.3: H^* is non-trivial with $a^*, b^* \in V(C)$.
In this case, we let H be a subgraph of G induced by $V(H^*) \cup \{x\}$. Note that a^*, b^*, c^* are vertices of attachment of H . Hence H is the desired C-flap of G and $(T_1^* \cup T_2^*)$ is also the desired $C\backslash(H\backslash\{a^*, b^*, c^*\}))$-Tutte subgraph of $G\backslash(H\backslash\{a^*, b^*, c^*\}$. (See Fig. 3.46.) ■

Finally, we use Lemma 3.5 to prove the following lemma.

Lemma 3.6 *Let G be 2-connected plane graph, C_1 be the outer walk of G, C_2 be*

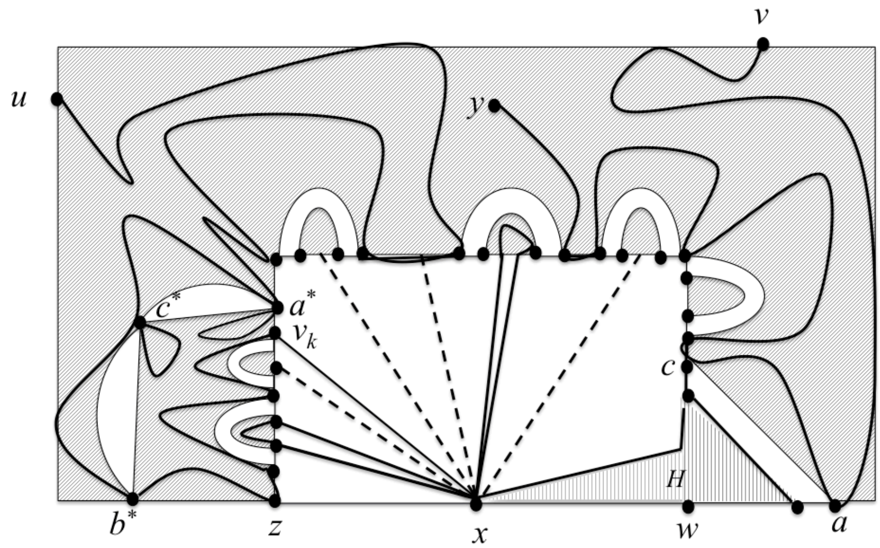


Figure 3.45: The structure of H, T_1' and T_2 in G in Case 2.2 when $w \notin V(T_1 \cup T_2)$.

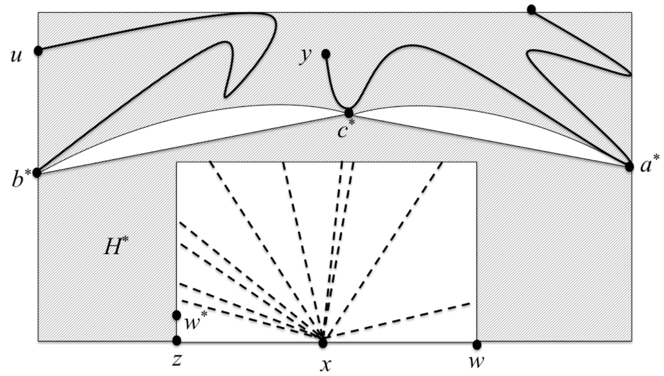


Figure 3.46: The structure of H, T_1^* and T_2^* in G in Case 2.2.

another facial cycle of G such that $V(C_1) \cap V(C_2) \neq \emptyset$, $x \in V(C_1)$, and $y \in V(G) \backslash \{x\}$.

Suppose that $w \in V(C_1)$ with $wx \in E(C_1)$. Then there exist:

(i) a C_1-flap H in G with attachments a, b, c such that $x \in (V(H) \backslash \{a\}) \cup \{b\}$, and if H is non-trivial, then $w \in V(H) \backslash \{b\}$ and a, w, x, b appear in $C_1 \cap H$ in this order;

(ii) a $((C_1 \cup C_2) \backslash (H \backslash \{a, b, c\}))$-Tutte trail T in $G \backslash (H \backslash \{a, b, c\})$ from b to y such that $a, c \in V(T)$, and T contains at least one vertex in $C_1 \cap C_2$.

(iii) Moreover, if every path from x to y contains a vertex of $C_1 \cap C_2$, then $H = \{x\}$ in (i) and (ii) above.

Proof. Let $k = |V(C_1 \cap C_2)|$.
Case 1: $k = 1$.
Note that in this case, G has a path from x to y which avoids $C_1 \cap C_2$ unless the vertex in $V(C_1) \cap V(C_2)$ is x or y. Let $V(C_1) \cap V(C_2) = \{d\}$ and let G_1 be a graph obtained by splitting d into two vertices d_0, d_1 as in Fig. 3.47. Note that G_1 is 2-connected.

First suppose that either $x = d$, or $y = d$. In this case, we construct H, T such that H is trivial. Assume without loss of generality that $x = d$. By Theorem 3.1(a), G_1 has an F_{G_1}-Tutte trail T' from d_0 to y containing d_1. Then $H = \{x\}$ is the desired C-flap, which satisfies (iii), and, by identifying d_0 and d_1, T' will become the desired $(C_1 \cup C_2)$-Tutte trail T of G from x to y.

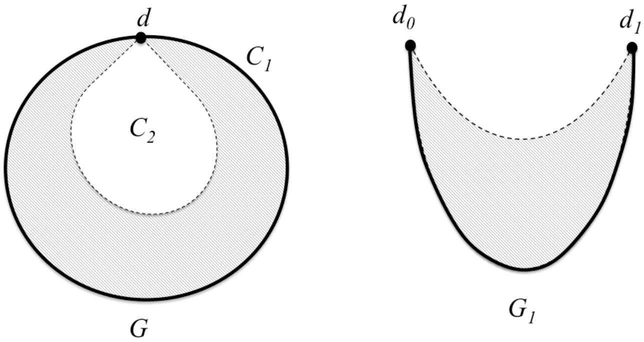


Figure 3.47: The structure of G and G_1 in Case 1.

Next, suppose that $x \neq d$ and $y \neq d$. We let $w^* = d_i$ if $w \in V(C_1 \cap C_2)$ and $xd_i \in E(F_{G_1})$ for some $i = 0, 1$; otherwise, we let $w^* = w$. By Lemma 3.5(C2), there exist: an F_{G_1}-flap H in G_1 with attachments a, b, c such that $y, d_0, d_1 \notin V(H) \backslash \{a, b, c\}$ and $x \in (V(H) \backslash \{a\}) \cup \{b\}$, and if H is non-trivial, then $w^* \in V(H) \backslash \{b\}$ and a, w^*, x, b appear in $C \cap H$ in this order; a $(F_{G_1} \backslash (H \backslash \{a, b, c\}))$-Tutte subgraph in $G_1 \backslash (H \backslash \{a, b, c\})$ consisting of two edge-disjoint trails T_1 and T_2 such that $a, c \in V(T_1 \cup T_2)$, and T_1 and T_2 connect $\{b, y\}$ and $\{d_0, d_1\}$.(See Fig. 3.48.)

By identifying d_0 and d_1, H_1 will become the desired C_1-flap H in G, and $T_1 \cup T_2$ will become the desired $((C_1 \cup C_2) \backslash (H \backslash \{a, b, c\}))$-Tutte trail T in $G \backslash (H \backslash \{a, b, c\})$

from b to y.

Case 2: $k \geq 2$.

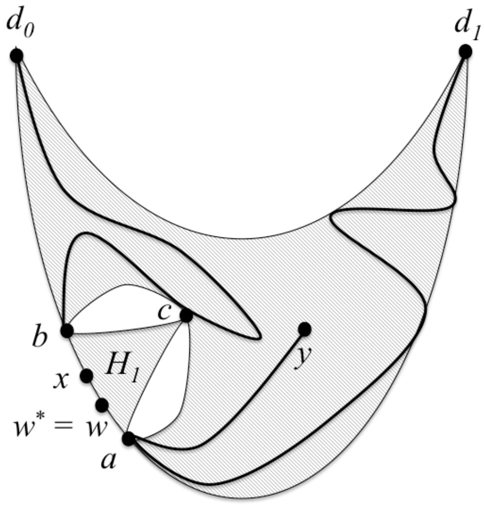


Figure 3.48: The structure of H_1, T_1 and T_2 of G_1 in Case 1.

Then we can express G as $G = G_1 \cup G'$ where $V(G_1) \cap V(G') = \{a_0, a_1\} \subseteq V(C_1) \cap V(C_2)$, $x \in V(G_1) \backslash \{a_1\}, wx \in E(G_1)$, and either G is 2-connected or $V(G_1) = \{w, x\}$. Then $G' = G_S$ where S is a chain of blocks $a_1, G_2, a_2, ..., a_{k-1}, A_k, a_k = a_0$. (See Fig. 3.49.) Note that $C_1 \cup C_2 = \bigcup_{i=1}^{k} F_{G_i})$. By Theorem 3.1(a), G_i has an F_{G_i}-Tutte trail T_{G_i} from a_{i-1} to a_i for all $1 \leq i \leq k$.

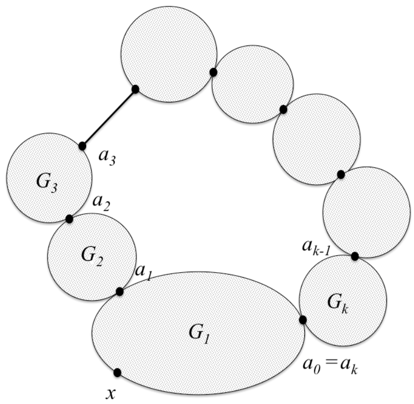


Figure 3.49: The structure of G in Case 2.

Case 2.1: $y \in V(G_m)\backslash\{a_{m-1}\}$ for some $2 \leq m \leq k$.
In this case, we must show that H is trivial since every path from x to y will contain a vertex of $C_1 \cap C_2$. By Theorem 3.1(a), G_1 has an F_{G_1}-Tutte trail T'_{G_1} from x to a_1 containing a_0, and G_m has an F_{G_1}-Tutte trail T'_{G_m} from a_{m-1} to y containing a_m. We let $s = a_0$, $t = a_m$, and if $m < k$, let $D = \bigcup_{i=m+1}^{k} G_i$; otherwise $D = \{s\} = \{t\}$. Then we will define trails T_s and T_t as follows.

- When $D = \{s\}$, we let $T_s = T_t = \{s\}$.
- When D is 2-edge-connected, we let $T_t = \{t\}$. By Theorem 3.1, we can choose T_s to be an F_D-Tutte trail of D from s to s containing t.
- When D is not 2-edge-connected, we let P_s (respectively, P_t) be an edge-block of D containing s(respectively, t). Let v_s(respectively, v_t) be the vertex of P_s(respectively, P_t) such that $v_s \neq s$(respectively, $v_t \neq t$) and v_s(respectively, v_t) has a neighbor in $D\backslash P_s$(respectively, $D\backslash P_t$). By Theorem 3.1, P_s has an F_{P_s}-Tutte trail T_s from s to s containing v_s, and P_t has an F_{P_t}-Tutte trail T_t from t to t containing v_t.

In each case, $H = \{x\}$ is the desired C_1-flap in G and $T = T'_{G_1} \cup T'_{G_m} \cup T_s \cup T_t \cup \bigcup_{i=2}^{m-1} T_{G_i}$ is the desired $(C_1 \cup C_2)$-Tutte trail in G from x to y. (See Fig. 3.50.)

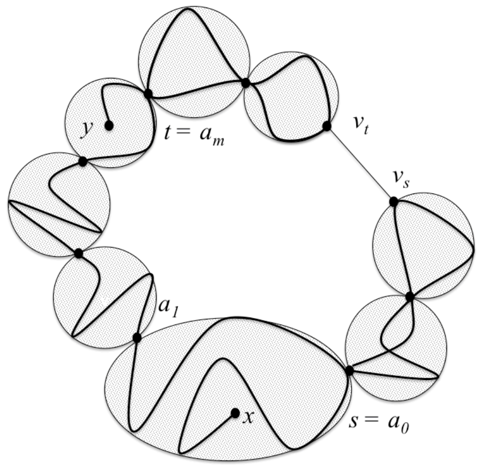


Figure 3.50: The structure of T in Case 2.1.

Case 2.2: $y \in V(G_1)$ and $x = a_0$.
In this case, we must show that H is trivial since $x \in V(C_1 \cap C_2)$. By Theorem

3.1(a), G_1 has an F_{G_1}-Tutte trail T'_{G_1} from x to y containing a_0. We let $t = a_1$ and $D = \bigcup_{i=2}^{k} G_i$. Then we use the same proof as for Case 2.1 to define T_x and T_t in D. Hence $H = \{x\}$ is the desired C_1-flap in G and $T = T'_{G_1} \cup T_s \cup T_t$ is the desired $(C_1 \cup C_2)$-Tutte trail in G from x to y. (See Fig. 3.51.)

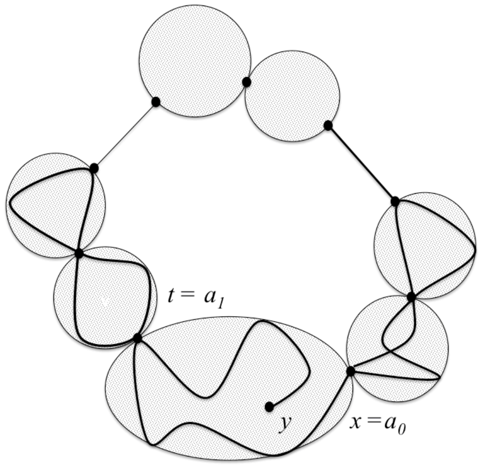


Figure 3.51: The structure of T in Case 2.2.

Case 2.3: $y \in V(G_1)$ and $\{x, y\} \cap \{a_0, a_1\} = \emptyset$.
Since $wx \in V(G_1)$, by Lemma 3.5(C2), there exist: an F_{G_1}-flap H in G_1 with attachments a, b, c such that $y, a_0, a_1 \notin V(H)\backslash\{a, b, c\}$ and $x \in (V(H)\backslash\{a\}) \cup \{b\}$, and if H is non-trivial, then $w \in V(H)\backslash\{b\}$ and a, w, x, b appear in $C \cap H$ in this order; an $(F_{G_1}\backslash(H\backslash\{a, b, c\}))$-Tutte subgraph in $G_1\backslash(H\backslash\{a, b, c\})$ consisting of two edge-disjoint trails T_1 and T_2 such that $a, c \in V(T_1 \cup T_2)$, and T_1 and T_2 connect $\{b, y\}$ and $\{u, v\}$.

Then H is the desired C_1-flap of G and $T = T_1 \cup T_2 \bigcup_{i=2}^{k} T_{G_i}$ is the desired $((C_1 \cup C_2)\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$ from x to y. (See Fig. 3.52.) ■

Our next result will be used for finding a Tutte closed trail in toroidal graphs.

Theorem 3.7 *Let G be a 2-connected plane graph, let C be the outer walk of G, let $u_1, v_1 \in V(C)$ and $u_2, v_2 \in V(G)\backslash\{u_1, v_1\}$ be such that $u_1 \neq v_1$ and there is a path in $G\backslash C[u_1, v_1]$ from u_2 to v_2. Then there exists a $C_1[u_1, v_1]$-Tutte subgraph of G consisting of two edge-disjoint trails T_1 from u_1 to v_1, and T_2 from u_2 to v_2.*

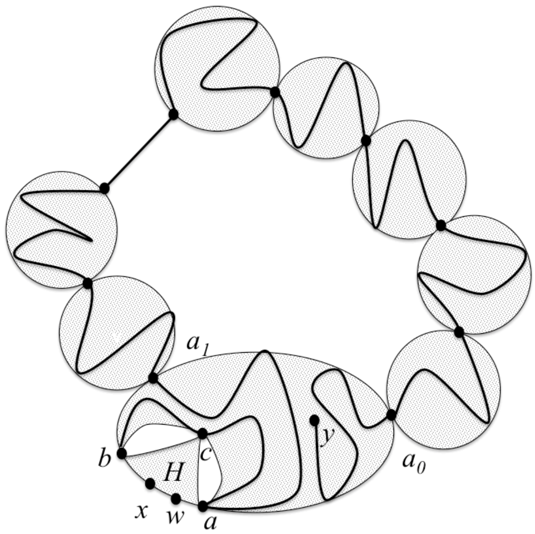


Figure 3.52: The structure of H and T in Case 2.3.

Proof. We prove the theorem by using induction on $|V(G)|$. If $|V(G)| = 3$, we take $T_1 = \{u_1, v_1, u_1v_1\}$ and $T_2 = \{u_2\} = \{v_2\}$ and then $T_1 \cup T_2$ is a spanning subgraph of G. So we assume $|V(G)| \geq 4$ and proceed to the induction step.

Case 1: G has a 2-separation (G_1, G_2) such that $V(G_1) \cap V(G_2) \subseteq V(C[u_1, v_1])$, $E(G_1) \cap E(C[v_1, u_1]) \neq \emptyset$, $E(G_2) \cap E(C[v_1, u_1]) = \emptyset$, and $E(G_2) \cap E(C[u_1, v_1]) \neq \emptyset$.

Let $z_1, z_2 \in V(G_1) \cap V(G_2)$ such that $C[z_1, z_2] \subseteq G_2$. Suppose (G_1, G_2) is chosen such that $|V(G_2)|$ is as small as possible. Let G_1^* be a graph obtained from G_1 by adding an edge $e = uv$ such that $F_{G_1^*} - e \subseteq C$. (See Fig. 3.53.) Note that G_1^* is 2-connected.

Case 1.1: $u_2, v_2 \in V(G_1)$.

By induction, G_1^* has an $F_{G_1^*}[u_1, v_1]$-Tutte subgraph T^* consisting of two edge-disjoint trails T_1^* from u_1 to v_1, and T_2^* from u_2 to v_2

Case 1.1.1: $e \in E(T^*)$.

By Corollary 3.2, G_2 has an F_{G_2}-Tutte trail T_{G_2} from z_1 to z_2. Then $(T^* - e) \cup T_{G_2}$ is the desired $C[u_1, v_1]$-Tutte subgraph of G. (See Fig. 3.54.)

Case 1.1.2: $z_1, z_2 \notin V(T^*)$.

Note that the component of $G_1^* \backslash V(T^*)$ containing z_1 and z_2 has two edges connecting it to T^*. Then T^* is the desired $C[u_1, v_1]$-Tutte subgraph of G. (See Fig. 3.55.)

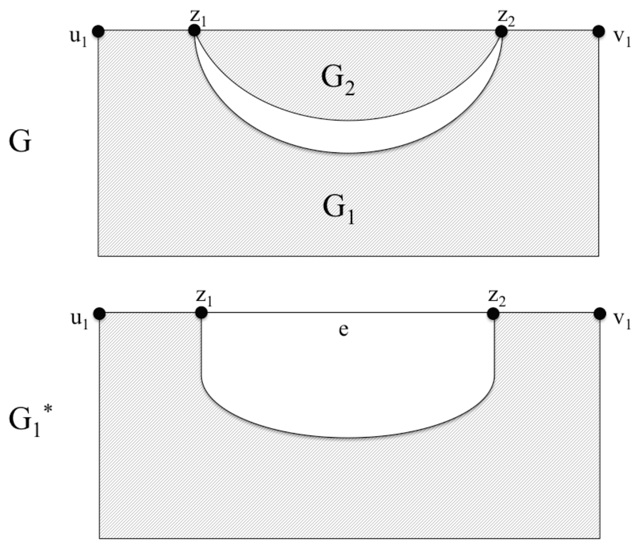


Figure 3.53: The structure of G and G_1^* in Case 1.

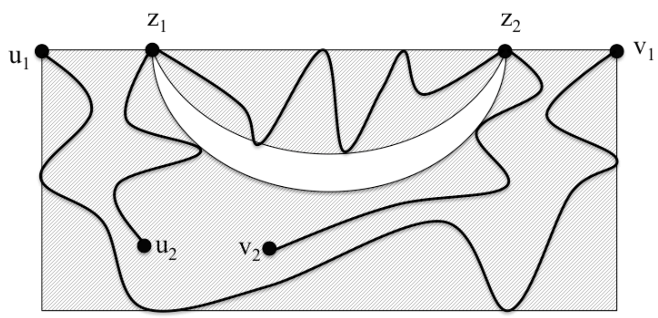


Figure 3.54: The structure of G in Case 1.1 when $e \in E(T^*)$.

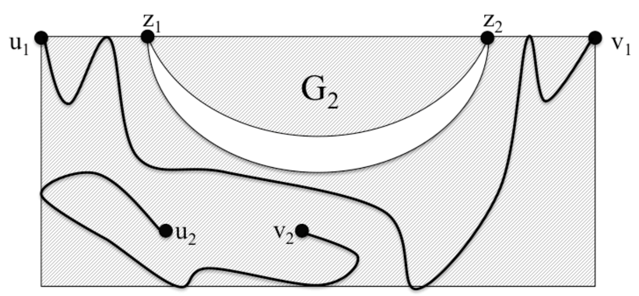


Figure 3.55: The structure of G in Case 1.1 when $z_1, z_2 \notin V(T^*)$.

Case 1.1.3: $\{z_1, z_2\} \cap V(T^*) \neq \emptyset$ and $e \notin V(T^*)$.
Assume without loss of generality that $z_1 \in V(T^*)$.

Suppose $z_2 \notin V(T^*)$. Then the component D of $G_1^* \backslash V(T^*)$ containing z_2 has two edges e, f connecting it to T^*. Let w be an end vertex of f in T^*, let $D^* = (D \backslash e) \cup G_2$, and let D_1 be an edge-block of D^* containing z_1. If $|V(D_1)| \geq 2$, let $d_1 \in V(D_1) \backslash \{z_1\}$ such that d_1 has a neighbor in $V(G \backslash D_1)$. (If $|V(D_1)| = 1$, we let $z_1 = d_1$.) By Theorem 3.1(a), D_1 has an F_{D_1}-Tutte trail T_1 from z_1 to z_1 containing d_1. Then $T^* \cup T_1$ is the desired $C[u_1, v_1]$-Tutte subgraph of G. (See Fig. 3.56.)

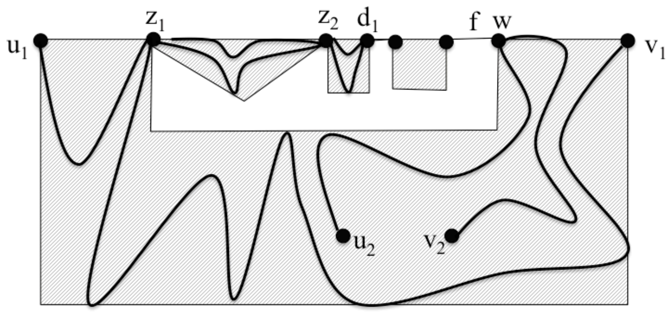


Figure 3.56: The structure of G in Case 1.1 when $z_1 \in V(T^*)$ and $e \notin V(T^*)$.

Suppose $z_2 \in V(T^*)$ and G_2 is 2-edge-connected. By Theorem 3.1(a), G_2 has an F_{G_2}-Tutte trail T_1 from z_1 to z_1 containing z_2. Then $T^* \cup T_1$ is the desired $C[u_1, v_1]$-Tutte subgraph of G.

Suppose $z_2 \in V(T^*)$ and G_2 is not 2-edge-connected. Since $|V(G_2)|$ is minimal, $V(G_2) = \{z_1, y, z_2\}$ and $e_1 = z_1y$ and $e_2 = z_2y$ are in $E(G_2)$. We will define a trail T_1 as follows.
If $E(G_2)$ has only two edges, we choose $T_1 = \{z_1\}$.
If $e_1' \in E(G_2)$ where $e_1' = z_1y$ is a parallel edge to e_1 , we choose $T_1 = \{z_1, y, e_1, e_1'\}$.
If $e_2' \in E(G_2)$ where $e_2' = z_2y$ is a parallel edge to e_2 , we choose $T_1 = \{z_2, y, e_2, e_2'\}$.
Then $T^* \cup T_1$ is the desired $C[u_1, v_1]$-Tutte subgraph of G.

Case 1.2: $u_2, v_2 \in V(G_2)$.
Since $|V(G_2)|$ is minimal and there is a path in $G \backslash C[u_1, v_1]$ from u_2 to v_2, G_2 is 2-connected. By induction, G_2 has a $C[z_1, z_2]$-Tutte subgraph T consisting of two edge-disjoint trails T_1 from z_1 to z_2, and T_2 from u_2 to v_2. By Theorem 3.1(a), G_1^* has an $F_{G_1^*}$-Tutte trail T' from u_1 to v_1 containing e. Note that $T_1 \cup (T' - e)$ is a

trail in G from u_1 to v_1. Then $(T_1 \cup (T' - e)) \cup T_2$ is the desired $C[u_1, v_1]$-Tutte subgraph of G. (See Fig. 3.57.)

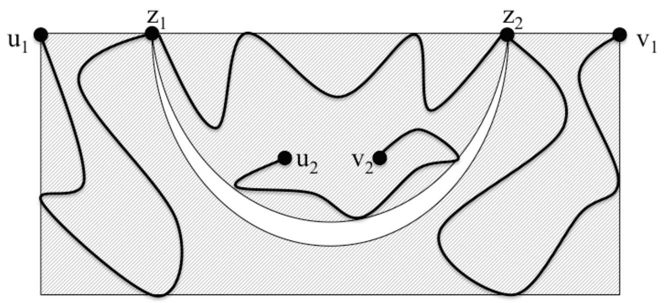


Figure 3.57: The structure of G in Case 1.2.

Case 2: G has no 2-separation as in Case 1.
Let $G^* = G \backslash (V(C[u_1, v_1]) \backslash \{u_1, v_1\})$. Note that since Case 1 does not occur, G^* is connected. By Corollary 3.2, there exists an F_{G^*}-Tutte trail T^* in G from u_2 to v_2 such that either $u_1 \in V(T^*)$ or the component M of $G^* \backslash V(T^*)$ containing u_1 has exactly one edge connecting it to T^*.

Case 2.1: Either $u_1 \in V(T^*)$, or $u_1 \notin V(T^*)$ and M does not contain v_1.(See Fig. 3.58.)

We will modify $C[u_1, v_1]$ by diverting it into each component J of $G \backslash (C[u_1, v_1] \cup T^*)$

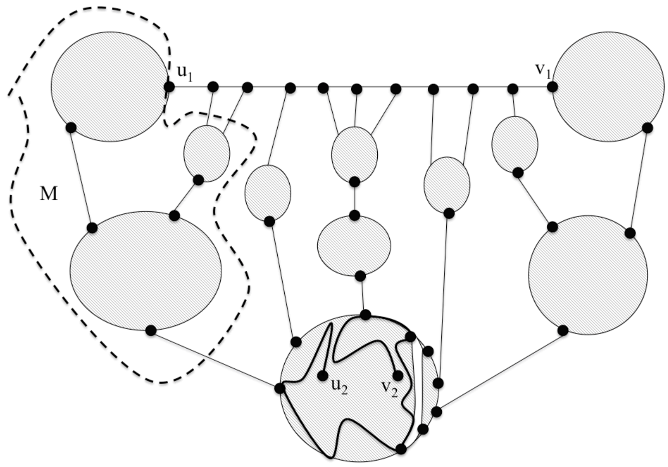


Figure 3.58: The structure of G in Case 2.1 when $u_1 \notin V(T^*)$

which has at least one neighbor on $C[u_1, v_1]$ to obtain the desired trail from u_1 to v_1. Note that each component J has at most two edges connecting it to T^* and at least one edge connecting it to $C[u_1, v_1]$. Let Q_J be the minimal path of $C[u_1, v_1]$ containing all vertices of attachment of J in $C[u_1, v_1]$. Let $J^* = J \cup Q_J$ (See Fig. 3.59.), a_J, b_J be end vertices of Q_J (Possibly, $a_J = b_J$.), and B_J be the edge-block of J^* containing Q_J. Let $c_1, c_2 \in V(B_J)$ such that c_1 and c_2 has a neighbor in $G \backslash B_J$. (Possibly, $c_1 = c_2$.) When J has at least two edges connecting it to $C[u_1, v_1]$, by Theorem 3.1, B_J has an F_{B_J}-Tutte trail T_J from a_J to b_J containing c_1. When J has only one edge connecting it to $C[u_1, v_1]$, Q_J has only one vertex and we let $T_J = Q_J$. Note that if $c_2 \notin V(T_J)$, the component of $G \backslash (T^* \cup T_J)$ containing c_2 has two edges connecting it to T_J and one edge connecting it to T^*. We replace Q_J by T_J for each such J. Let T' be the modified trail of $C[u_1, v_1]$. Then $T' \cup T^*$ is the desired

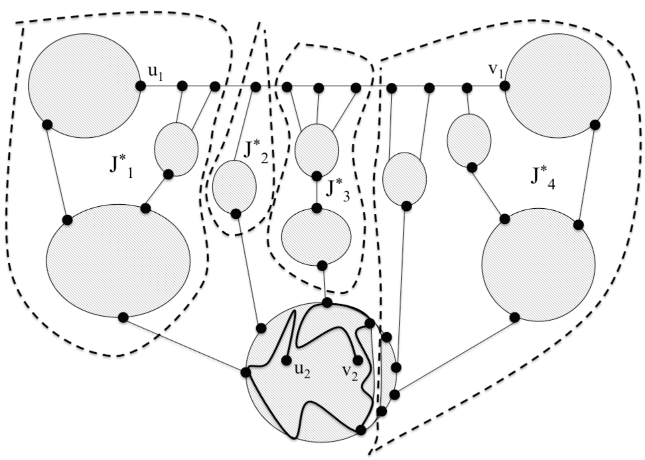


Figure 3.59: A subgraph $J_i^* = J_i \cup Q_{J_i}$ where J_i is a component of $G \backslash (C[u_1, v_1] \cup T^*)$ such that J has at least one neighbor on $C[u_1, v_1]$ in Case 2.1.

$C[u_1, v_1]$-Tutte subgraph of G. (See Fig. 3.60.)

Case 2.2: $u_1 \notin V(T^*)$ and M contains v_1. (See Fig. 3.61.)

Let $S = G^* \backslash M$. Then S is connected and contains u_2 and v_2. Let Q_S be a minimal path of $C[u_1, v_1]$ containing all neighbors of S in $C[u_1, v_1]$. Let s_1, s_2 be end vertices of Q_S such that $C[s_1, s_2] \subseteq C[u_1, v_1]$. Let $H = G \backslash (S \cup (Q_S \backslash \{s_1, s_2\}))$. Let H^* be a graph obtained from H by adding an edge $e = s_1 s_2$ such that $F_{H^*} - e = C \backslash (V(Q_S) \backslash \{s_1, s_2\})$. (See Fig. 3.62.) Since M contains both u_1 and v_1, u_1, v_1 and e are contained in the same edge-block B of H^*. (Possibly, $B = H^*$.) Note that $H^* \backslash B$

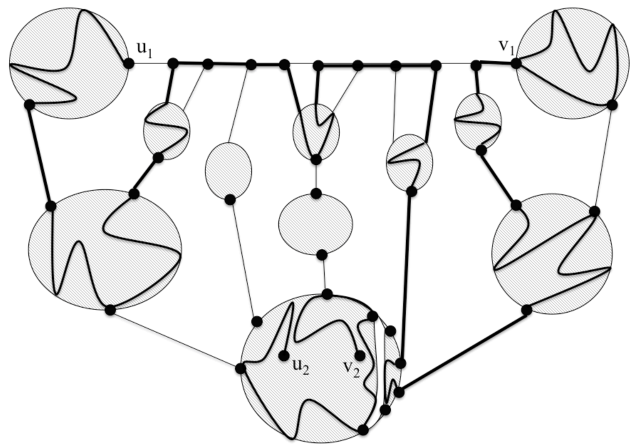


Figure 3.60: The structure of T' and T^* in Case 2.1.

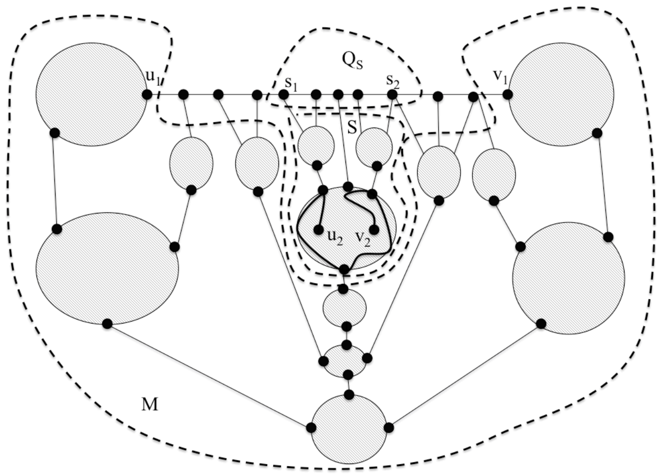


Figure 3.61: The structure of G in Case 2.2.

has at most one component. Let $h \in V(M)$ such that h has one edge connecting it to T^*. Note that if $H^* \neq B$, then $h \in V(H^* \backslash B)$. By Theorem 3.1(a), B has an F_B-Tutte trail T_B from u_1 to v_1 containing e. (See Fig. 3.62.) Note that if $h \notin V(T_B)$, then component H_1 of $H^* \backslash T_B$ containing h has at most two edges connecting it to T_B, and this implies that H_1 is a component of $G \backslash (T^* \cup (T_B - e))$ and has at most three edges connecting it to $T^* \cup (T_B - e)$.

We will modify $C[s_1, s_2]$ by diverting it into each component J of $G \backslash (C[s_1, s_2] \cup T^*)$ which has at least one neighbor on $C[s_1, s_2]$ to obtain the desired trail from s_1

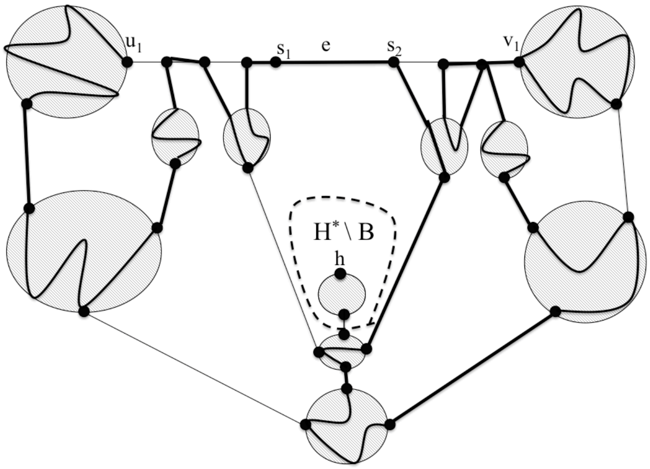


Figure 3.62: The structure of H^* and T_B in Case 2.2.

to s_2. Note that J has at most two edges connecting it to T^*. Then we use the same technique as in Case 2.1 to get T_J. We replace Q_J by T_J for each such J. Let T'' be the modified trail of $C_1[u_1, v_1]$. Since $T'' \cup (T_B - e)$ is a trail from s_1 to s_2, $T = T'' \cup [T^* \cup (T_B - e)]$ is the desired $C[u_1, v_1]$-Tutte subgraph of G. (See Fig. 3.63.) ■

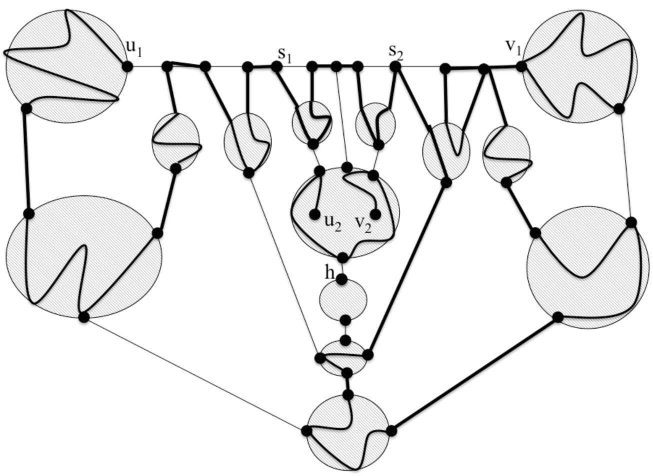


Figure 3.63: The structure of T in Case 2.2.

Chapter 4

Tutte Trails of Projective Plane Graphs

In this chapter, we use the concepts of face-width and representativity to analyse projective plane graphs. Recall from Chapter 2 that any two essential closed curves on the projective plane intersect.

The main result of this chapter is that every 2-edge-connected projective plane graph G has a Tutte trail from x to y for any $x, y \in V(G)$. We will prove this result when $x = y$ and $x \neq y$ in Section 4.1 and 4.2, respectively.

In Section 4.1, we prove Theorem 4.4 which tells us that every 2-edge-connected projective plane graph G has a C-Tutte closed trail containing e for any facial walk C of G, and $e \in V(C)$. This implies the main result when $x = y$ if we consider x as an end vertex of e.

In Section 4.2, we consider a 2-connected projective plane graph G with a facial walk C, $x \in V(C)$ and $y \in V(G)\backslash\{x\}$. In Theorem 4.7, we prove that there exist: a C-flap H of G with attachments $\{a, b, c\}$ such that $a, b \in V(C)$, $x \in V(H)$ and $y \notin V(H)\backslash\{a, b, c\}$; and a $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail of $G\backslash(H\backslash\{a, b, c\})$ from b to y containing {a, c}. Theorem 4.8 is the extension of Theorem 4.7 to 2-edge-connected graphs, and implies that every 2-edge-connected projective plane graph G has a Tutte trail from x to y for any distinct vertices x and y of G.

4.1 Tutte closed trails of projective plane graphs

If a projective plane graph G has representativity zero, then there is an essential closed curve ϕ which does not intersect G. This implies that there is no essential cycle in G.

We start this section by proving the following lemma.

Lemma 4.1 *Let G be a projective plane graph with representativity at least one. If there exists $x \in V(G)$ such that every essential cycle of G contains x, then the representativity of G is one.*

Proof. Since every essential cycle of G contains x, $G\backslash\{x\}$ has no essential cycle. Then there exists an essential closed curve α such that α does not intersect $G\backslash\{x\}$. Note that the representativity of $G\backslash\{x\}$ is zero. Then α can be redrawn so that it will intersect G only at x. So the representativity of G is one. ■

A closed curve is called *simple* if it does not cross itself. Let ϕ be a simple closed curve in a projective plane Σ. If ϕ is essential, $\Sigma\backslash\phi$ is homeomorphic to a disc. If ϕ is non-essential, then $\Sigma\backslash\phi$ is a disjoint union of surface Σ_1 and Σ_2 where Σ_1 is homeomorphic to a disc, and Σ_2 is homeomorphic to a projective plane. Since we can consider an essential cycle in a projective plane graph G as a simple essential closed curve in the projective plane, each of the faces of G are homeomorphic to a disc.

Next, we prove the following lemma.

Lemma 4.2 *Let G be a 2-connected projective plane graph with representativity at least two. Let (G_1, G_2) be a 2-separation of G such that G_1 contains an essential cycle, and subject to this condition, G_2 is minimal. Then G_2 does not contain an essential cycle.*

Proof. Let $x, y \in V(G_1) \cap V(G_2)$. Since G_1 contains an essential cycle, each of the faces of G are homeomorphic to a disc. Since $G_2\backslash\{x, y\}$ is connected (by the minimality of G_2), G_2 is contained in a face of G_1. (See Fig. 4.1.) Hence G_2 is contained in a disc so it does not contain an essential cycle. ■

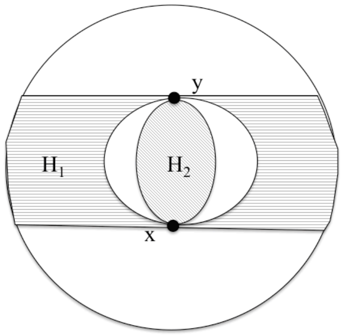


Figure 4.1: The structure of G_1 and G_2.

Recall from Chapter 2, let C be a cycle in a plane graph G, and $x, y \in V(C)$. $C[x, y]$ is a path of C from x to y in clockwise direction. Next, we show the following lemma.

Lemma 4.3 *Let G be 2-connected plane graph, F_G be the outer walk of G, $e \in E(F_G)$ and $v \in V(G) \backslash V(F_G)$. Then we can choose $f \in E(F_G) - e$ such that there is no 2-separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) \subseteq V(F_G)$, $E(G_2) \cap E(F_G) \neq \emptyset$, $e, f \in E(G_1)$ and $v \in V(G_2)$.*

Proof. Since G is 2-connected, we can choose two paths P_1, P_2 which intersect only at v from v to $V(F_G)$ and $|V(P_i) \cap V(F_G)| = 1$ for $i = 1, 2$. Let x and y be the end vertices of P_1 and P_2, respectively, on $V(F_G)$. Assume without loss of generality that e belongs to $E(F_G[x, y])$. So we choose $f \in E(F_G[y, x])$.

Suppose that there is a 2-separation (G_1, G_2) of G as in the statement of this lemma. Since $v \in V(G_2)$, $x, y \in V(G_2)$. (Possibly, $x \in V(G_1) \cap V(G_2)$ and/or $y \in V(G_1) \cap V(G_2)$.) Then either $F_G[x, y]$ or $F_G[y, x]$ are contained in G_2. Since $e \notin E(G_2)$, $F_G[x, y] \not\subseteq G_2$. This implies that $f \in E(F_G[y, x]) \subseteq E(G_2)$ which contradicts $f \in E(G_1)$. Hence there is no such 2-separation of G.

■

We use the ideas from [31] and Lemma 4.3 to prove the following theorem.

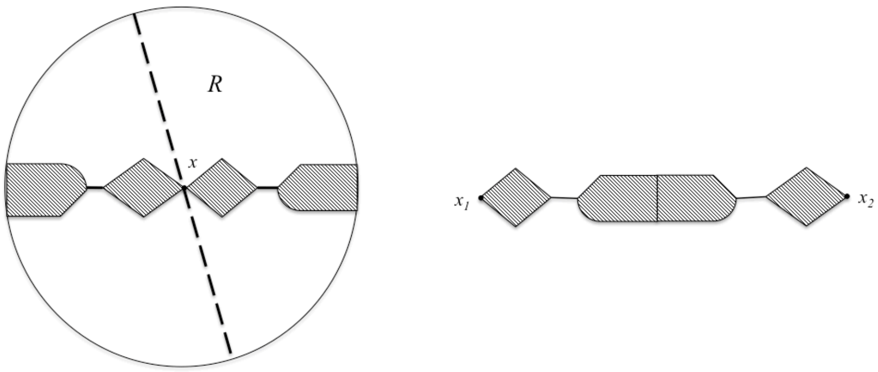
Theorem 4.4 *Let G be a 2-edge-connected graph embedded on the projective plane. Let R be a face of G, C be the facial walk of R and e be an edge of C. Then there* *is a C-Tutte closed trail T of G containing e such that every component of $G\backslash V(T)$ which contains an essential cycle is vertex-disjoint from C.*

Proof. If G doesn't have an essential cycle, then G is a plane graph and by Theorem 3.1, G has a desired C-Tutte trail containing e. Then we suppose the representativity of G is at least one. We prove the theorem by using induction on $|V(G)|$. If $|V(G) \leq 3$, there exists a spanning closed trail of G which contains e. So we assume $|V(G)| \geq 4$ and proceed to the induction step.

Case 1: G is 2-connected and the R-width of G is one.
Let $x \in V(C)$ and ϕ be an essential closed curve passing through R and intersecting G only at x. Let G' be the plane graph obtained by cutting G along ϕ, in which the vertices corresponding to x are x_1 and x_2. (See Fig. 4.2.) Then $G' = H_S$ where S is a chain of edge-blocks $x_1 = a_1, B_1, b_1a_2, B_2, ..., b_{k-1}a_k, B_k, b_k = x_2$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$. For each $1 \leq i \leq k$, we let F_i be an outer walk of B_i. By Theorem 3.1, B_i has an F_i-Tutte trail T_i from a_i to b_i. Let $T = T_1 \cup T_2 \cup ... \cup T_k \cup \{b_1a_2, b_2a_3, ..., b_{k-1}a_k\}$. If $e = b_ja_{j+1}$ for some $1 \leq j \leq k-1$, then we let U be a trail such that $U = T$. For the case $e \in E(F_l)$ for some $1 \leq l \leq k$, B_l has, by Theorem 3.1, an F_l-Tutte trail T_l'' from a_l to b_l containing e and we let $U = (T - T_l) \cup T_l''$. By identifying x_1 and x_2 in both cases, U becomes the desired C-Tutte closed trail U' of G. Since $x \in V(U')$, no component of $G\backslash V(U')$ contains an essential cycle.

Case 2: G is 2-connected and the R-width of G is two.
Note that C is a cycle.

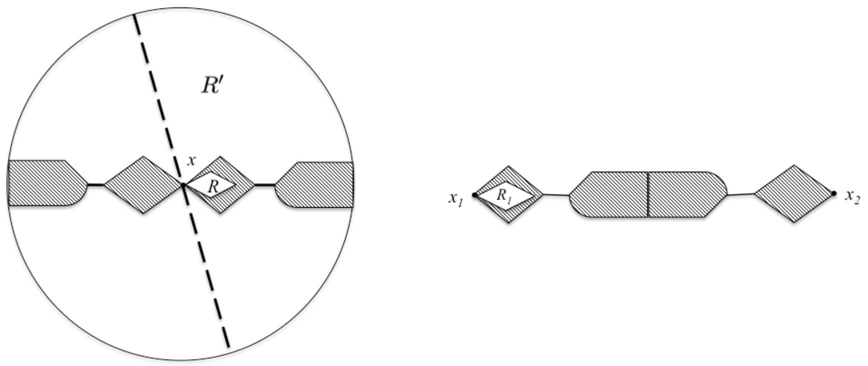
Case 2.1: There is a face R' of G such that the R'-width of G is one and $V(C') \cap V(C) \neq \emptyset$ where C' is the facial walk of R'.
The proof is similar to Case 1. Let $x \in V(C') \cap V(C)$ and ϕ be an essential closed curve passing through R' and intersecting G only at x. Let G' be the plane graph obtained by cutting G along ϕ, in which the vertices corresponding to x are x_1 and x_2 . We let R_1 be the face of G' corresponding to the face R of G and C'' be the facial walk of R_1. (See Fig. 4.3.) Then $G' = H_S$ where S is a chain of edge-blocks $x_1 = a_1, B_1, b_1a_2, B_2, ..., b_{k-1}a_k, B_k, b_k = x_2$, $C'' \subseteq B_1$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$. For each $2 \leq i \leq k$, we let F_i be an outer walk of B_i. By Theorem 3.1,



G $\qquad$ G'

Figure 4.2: The structure of G and G' in Case 1.

B_1 has a C''-Tutte trail T_1 from x_1 to b_1 containing e and B_i has an F_i-Tutte trail T_i from a_i to b_i for all $2 \leq i \leq k$. Let $T = T_1 \cup T_2 \cup ... \cup T_k \cup \{b_1a_2, b_2a_3, ..., b_{k-1}a_k\}$. By identifying x_1 and x_2, T becomes the desired C-Tutte closed trail T' of G. Since $x \in V(T')$, no component of $G \backslash (T')$ contains an essential cycle.



G $\qquad$ G'

Figure 4.3: The structure of G and G' in Case 2.1.

Case 2.2: There is no face as in Case 2.1.
Let α be an essential closed curve passing through R and intersecting G at vertices a and b. Let G^* be the plane graph obtained by cutting G along α, in which the

vertices corresponding to a are a_1 and a_2, and the vertices corresponding to b are b_1 and b_2. Let G' be a new graph obtained from G^* by adding edges a_1b_2, a_2b_1 such that these edges are on the outer walk C_1 of G'. (See Fig. 4.4.) Note that $e \in E(F_{G'})$. Then G' is 2-connected and, by Theorem 3.1, has a C_1-Tutte closed trail T containing a_1b_2, a_2b_1 and e. By identifying a_1 and a_2, and b_1 and b_2, T becomes the desired C-Tutte closed trail T' of G. Since $a, b \in V(T')$ and every essential cycle must contain either a or b, no component of $G \backslash V(T')$ contains an essential cycle.

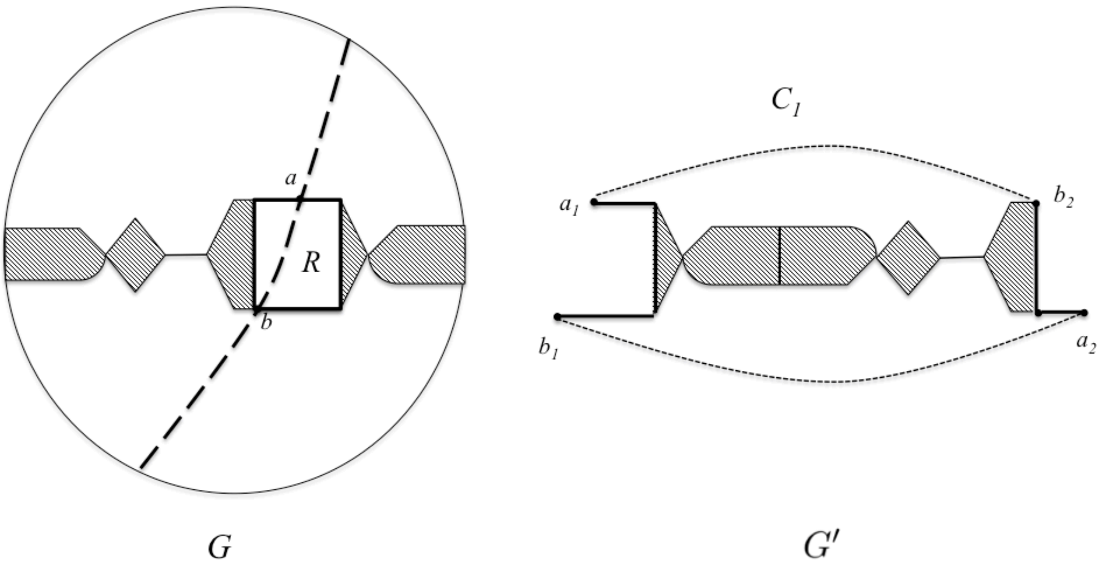


Figure 4.4: The structure of G and G' in Case 2.2.

Case 3: G is 2-connected and the R-width of G is at least three and $G \backslash V(C)$ has at least two blocks containing an essential cycle.

Since $G \backslash V(C)$ has an essential cycle, the representativity of $G \backslash V(C)$ is at least one. Let B_1 and B_2 be distinct blocks of $G \backslash V(C)$ containing an essential cycle. Let C_1 and C_2 be an essential cycle of B_1 and B_2, respectively. Since $V(C_1) \cap V(C_2) \neq \emptyset$ and $|V(B_1) \cap V(B_2)| \leq 1$, $|V(B_1) \cap V(B_2)| = 1$ and we let $v \in V(B_1) \cap V(B_2)$. (See Fig. 4.5.) If $G \backslash V(C)$ has at least three blocks containing an essential cycle, we let B_3 be block of $G \backslash V(C)$ containing an essential cycle, $B_3 \neq B_1$ and $B_3 \neq B_2$. Since every essential cycle must intersect, we let $v_1 \in V(B_1) \cap V(B_3)$ and $v_2 \in V(B_2) \cap V(B_3)$. If

v, v_1 and v_2 are not the same vertex, then $B_1 \cup B_3$, $B_2 \cup B_3$ or $B_1 \cup B_2 \cup B_3$ are blocks of $G \backslash V(C)$ which contradict the maximality of B_1, B_2 or B_3. Then $v = v_1 = v_2$ and this implies that every blocks of $G \backslash V(C)$ containing an essential cycle must contains v. By Lemma 4.1, the representativity of $G \backslash V(C)$ is one.

Then G can be re-embedded in the plane such that C is the outer cycle. (See Fig. 4.6.) By Lemma 4.3, we can choose $f \in E(C)$ such that there is no 2-separation (G_1, G_2) of G with $V(G_1) \cap V(G_2) \subseteq V(C), E(G_2) \cap E(C) \neq \emptyset, e, f \in V(G_1)$ and $v \in V(G_2)$. By Theorem 3.1, there is a C-Tutte closed trail T containing e and f. Let B be a component of $G \backslash V(T)$ containing a vertex of $V(C)$. Then B has exactly two edges connecting it to T, and both edges have end vertices in $V(T) \cap V(C)$. Since there is no such 2-separation as in Lemma 4.3 ,$v \notin V(B)$ and B does not contain an essential cycle.

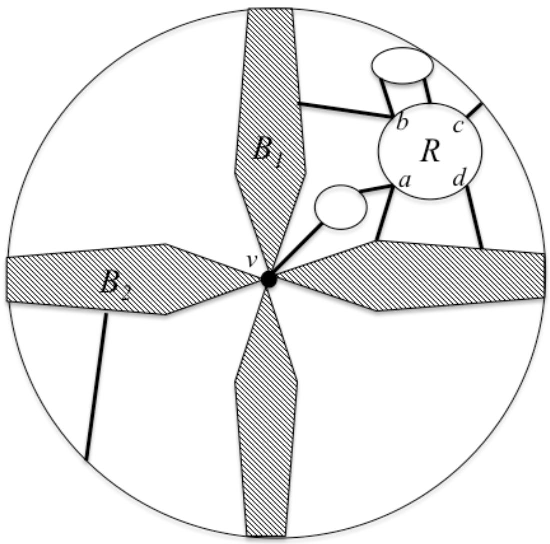


Figure 4.5: The structure of blocks B_1, B_2 of $G \backslash V(C)$ in Case 3 which contain an essential cycle.

Case 4: G is 2-connected and the R-width of G is at least three and $G \backslash V(C)$ has only one block containing an essential cycle.
We assume that C is traversed in a fixed direction, which we call the "clockwise direction" of C. We refer to the reverse trajectory of C as the "counter-clockwise" direction of C.

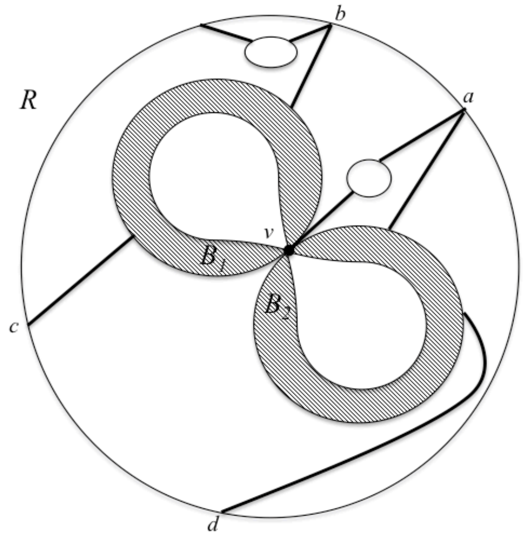


Figure 4.6: The plane graph G from Fig. 4.5 re-embedded in the plane with R as the outer face.

Let H'' be the block of $G\backslash V(C)$ containing an essential cycle, H be the edge-block of $G\backslash V(C)$ containing H'', R_1 be the face of H containing R, and F be the facial walk of R_1. (See Fig. 4.7.) Let X be a $(C \cup H)$-bridge of G. Since G is 2-connected, X has vertices of attachment in C. We let Q_X be the minimal path in C including all vertices of attachment of X in C such that no interior vertex of Q_X shares a face with a vertex of H. (Possibly Q_X is a single vertex.) Let p_X and q_X be the end vertices of Q_X, respectively. Note that for two $(C \cup H)$-bridges X, Y, either $Q_X \subset Q_Y, Q_Y \subset Q_X$, or $E(Q_X) \cap E(Q_Y) = \emptyset$. A $(C \cup H)$-bridge X is *maximal* if $|V(Q_X)| = 1$, or there is no $(F \cup H)$-bridge Y such that $Q_X \subset Q_Y$.

A $(C \cup H)$-*bridge group* A is the union of a maximal $(C \cup H)$-bridge X together with all $(C \cup H)$-bridges Y such that $|V(Q_Y)| \geq 2$ and $Q_Y \subset Q_X$. We put $Q_A = Q_X, p_A = p_X$ and $q_A = q_X$ where X is the maximal $(C \cup H)$-bridge in A.

A (C, H)-*connector* in G is a bridge of $C \cup H$ in G with its vertices of attachment in both C and H. A (C, H)-*connector group* M is an $(C \cup H)$-bridge group which contains an (C, H)-connector in G. Moreover, M is called trivial if $|V(M)| = 2$.

For each (C, H)-connector group M of G, let v_M be the unique vertex of attachment of M in H. Let K be the (C, H)-connector group such that $e \in E(Q_K)$, or

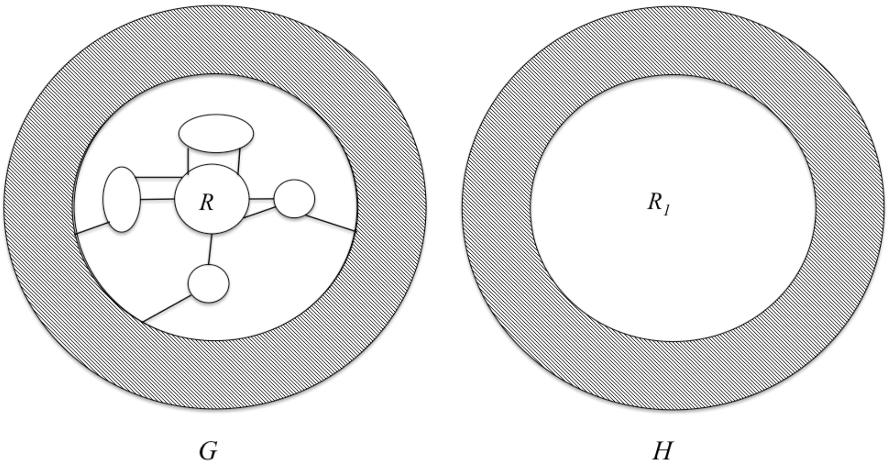


Figure 4.7: The structure of G and H in Case 4.

if there is no such group, the (C, H)-connector group with q_K nearest to e counter-clockwise from it, and such that K shares a face with either the edge e or the $(F_G \cup H)$-bridge group A with $e \in E(Q_A)$.

Let L be the (C, H)-connector group such that q_L is nearest counter-clockwise to p_K (possibly $q_L = p_K$) and q_L, v_L, p_K, e_K all lies on the same face R_2 of G, where e_K is the edge of K containing v_K. We also let F_2 be the facial walk of R_2. (See Fig. 4.8.)

Let P_1 and P_2 be the paths in C from p_L counter-clockwise to q_K and the path in C from q_L clockwise to p_K, respectively.

Next, we will construct a trail T' in H as follows.
If $v_K = v_L$, then, by induction, H has an F-Tutte closed trail T' containing an edge of F incident to v_K such that every component of $H \backslash V(T')$ containing a vertex of F' does not contain an essential cycle.
For the case $v_K \neq v_L$. we let $H_1 = H \cup \{v_K v_L\}$ be such that $v_K v_K$ is embedded in the face R_2. Let R_3 be the face of H_1 which contains R and F_3 be the facial walk of R_3. (See Fig. 4.8.) Then H_1 is 2-connected and $v_K v_L \in E(F_3)$. By induction, H_1 has an F_3-Tutte closed trail T'' containing $v_K v_L$ such that every component of $H_1 \backslash V(T'')$ containing a vertex of F' does not contain an essential cycle. Let $T' = T'' - \{v_K v_L\}$. In both cases no component of $G \backslash V(T')$ which contains a vertex of C can contain an essential cycle. Note that we are using the same idea as we used for plane graphs

in the proof of Theorem 3.1.

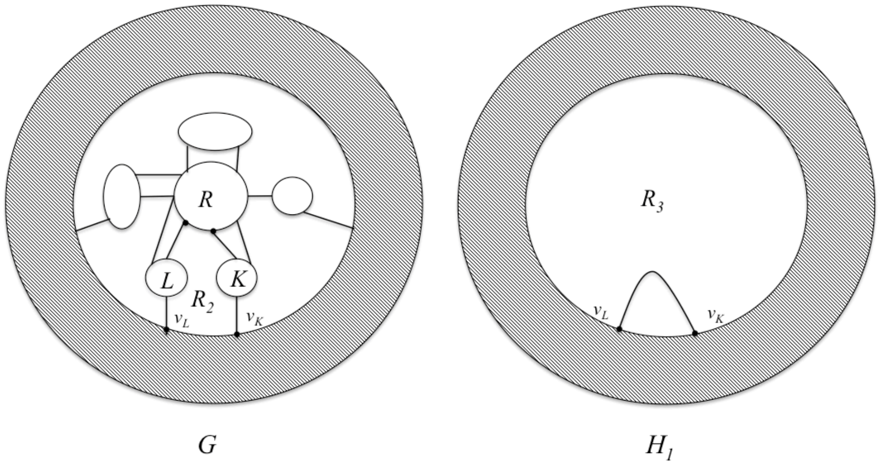


Figure 4.8: The structure of G and H_1 in Case 4.

Next, we consider K as a plane graph and define a Tutte trail T_K of K as follows. If K is a trivial (C, H)-connector group, we let $T_K = K$. For the case when K is a non-trivial (C, H)-connector group, we let $e_1 = v_K q_K$ and $M = K \cup Q_K \cup \{e_1\}$ with e_1 embedded in the outer walk such that $V(Q_K) \subset V(F_M)$ and we let d be an edge of F_M containing p_K. If $e \notin E(Q_K)$, then, by Theorem 3.1(b, c), M has an F_M-Tutte closed trail C containing d and e_1. For the case $e \in E(Q_K)$, by Theorem 3.1(b), M has an F_M-Tutte closed trail C containing d, e_1 and e (possibly $d = e$). In both cases, we let $T_K = C - e_1$.

Next, we also consider L as a plane graph and define a Tutte trail T_L of L in the similar way to T_K. If L is a trivial (F_G, H)-connector group, we let $T_L = L$. For the case when L is a non-trivial (F_G, H)-connector group, we let $e_2 = v_L p_L$ and $N = L \cup Q_L \cup \{e_2\}$ with e_2 embedded in the outer walk such that $V(Q_L) \subset V(F_N)$ and we let d' be an edge of F_N containing q_L. By Theorem 3.1(b, c), N has an F_N-Tutte closed trail C' containing d' and e_2. Let $T_L = C' - e_2$.

Finally, we let $T = T' \cup T_K \cup T_L \cup P_1$.

Next, we will modify T by diverting it into the $(P_1 \cup T')$-bridge groups J of G such that J has vertices of attachment on P_1 to obtain the desired C-Tutte closed trail containing e.

Let J be a $(P_1 \cup T')$-bridge group of G such that $J \neq K$, $J \neq L$, J has vertices of attachment on P_1, and J has more than three edges of attachment on $P_1 \cup T'$. By construction of H and T', J has at most two vertices of attachment on T'. Then we can modify J in three cases depend on the number of vertices of attachment on T' using the same method as in Subcase 3.1, 3.2 or 3.3 of the proof of Theorem 3.1.

We have now constructed a trail T containing e. If P_2 is a single vertex, then T is the desired C-Tutte closed trail in G. So suppose $V(P_2) \geq 2$ and let J' be the T-bridge in G containing P_2, and J^* be the graph obtained from J' by adding an edge e_{J^*} between q_L and p_K such that $e_{J^*} \in E(F_{J^*})$ and $F_{J^*} - e_{J^*} = P_2$. Then, by Theorem 3.1(a), J^* has F_{J^*}-Tutte trail T_{J^*} from q_L to p_K containing e_{J^*}. Hence $T \cup (T_{J^*} - e_{J^*})$ is the desired C-Tutte closed Tutte trail containing e in G.

Case 5: G has a 1-separation (G_1, G_2).
Let $V(G_1) \cap V(G_2) = \{x\}$. Note that G_1 and G_2 are 2-edge-connected. Assume without loss of generality that G_1 has an essential cycle.

Case 5.1: G_2 does not contain an essential cycle. (See Fig. 4.9.)
We define a facial walk C_i of a face R_i of G_i for $i = 1, 2$ as follows.
(i) If $C \not\subseteq G_1$ and $C \not\subseteq G_2$, then $x \in V(C)$ and for $i = 1, 2$, we let $C_i = C \cap G_i$.
(ii) If $C \subseteq G_1$, we let $C_1 = C, R_1 = R$ and C_2 be a facial walk containing an edge incident to x.
(iii) If $C \subseteq G2$, we let $C_2 = C, R_2 = R$ and C_1 be a facial walk containing an edge incident to x.

Subcase 5.1.1: $e \in E(G_1)$.
By induction, G_1 has a C_1-Tutte closed trail T_1 containing e such that every component of $G_1 \backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C_1. If $x \notin V(T_1)$, then T_1 is the desired C-Tutte closed trail in G. Hence suppose $x \in V(T_1)$. Since G_2 is a plane graph, by Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2 from x to x. Then $T_1 \cup T_2$ is the desired C-Tutte closed trail of G.

Subcase 5.1.2: $e \in E(G_2)$.
By Theorem 4(a), G_2 has a C_2-Tutte trail T_2' from x to x containing e. Then by induction, G_1 has a C_1-Tutte closed trail T_1' containing an edge of C_1 incident to x such that every component of $G_1 \backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C_1. Then $T_1' \cup T_2'$ is the desired C-Tutte closed trail of G.

Case 5.2: G_2 contains an essential cycle. (See Fig. 4.10 and 4.11.)

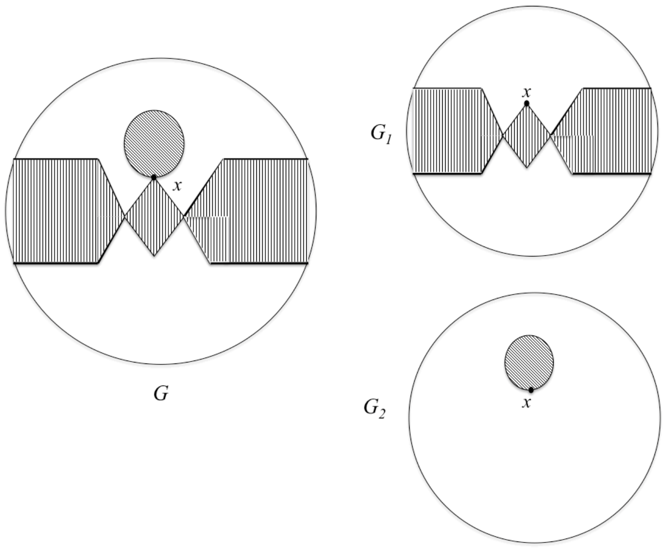


Figure 4.9: The structure of G, G_1 and G_2 in Case 5.1.

Suppose that there is no 1-separation (H_1, H_2) of G such that either H_1 or H_2 does not contains an essential cycle. Note that the representativity of G is one and every essential cycle of G contains x. Assume without the loss of generality that $e \in E(G_1)$. Let R' be a face of G such that R'-width is one. (If R-width is one, we let $R = R'$.) If G_1 is not 2-connected, then there is a 1-separation (H'_1, H'_2) of G_1 such that H'_1 is 2-connected. Thus $(H'_1, H'_2 \cup G_2)$ is a 1-separation of G. Hence we can assume that G_1 is 2-connected. For $i = 1, 2$, we let R'_i be the face of G_i such that R_i-width is one and $R' \subseteq R'_i$. Let C'' be the facial walk of R'_2 and f be an edge of C'' containing x. By induction, G_2 has the desired C''-Tutte closed trail T_2 containing f.

Subcase 5.2.1: The R-width of G is one. (See Fig. 4.10.)
Let C^* be the facial walk of R'_1 of G_1. Note that $C \subset C^*$. Then we use the same proof as Case 1. Thus G_1 has the desired C^*-Tutte closed trail T_1 containing e and x. Then $T_1 \cup T_2$ is the desired C-Tutte closed trail of G.

Subcase 5.2.2: The R-width of G is at least two. (See Fig. 4.11.)
Let ϕ be an essential closed curve passing through R'_1 and intersecting G_1 only at x. Let G'_1 be the plane graph obtained by cutting G_1 along ϕ, in which the vertices corresponding to x are x_1 and x_2. We let R'' be the face of G'_1 corresponding to R of G and C' be the facial walk of R''. Then $G'_1 = H_S$ where S is a chain of

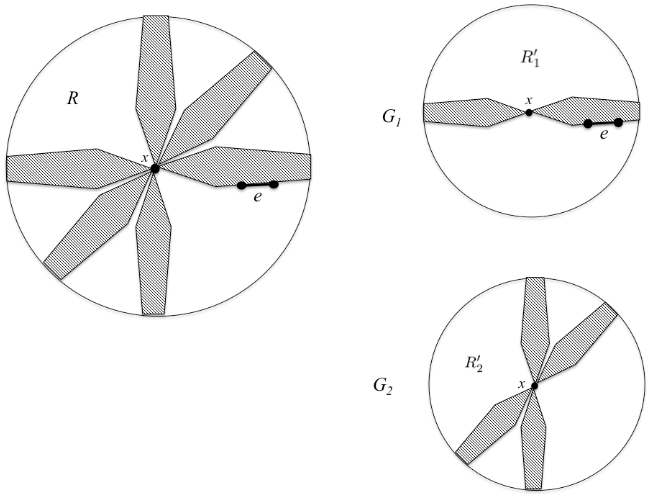


Figure 4.10: The structure of G, G_1 and G_2 in Case 5.2 when $R = R'$.

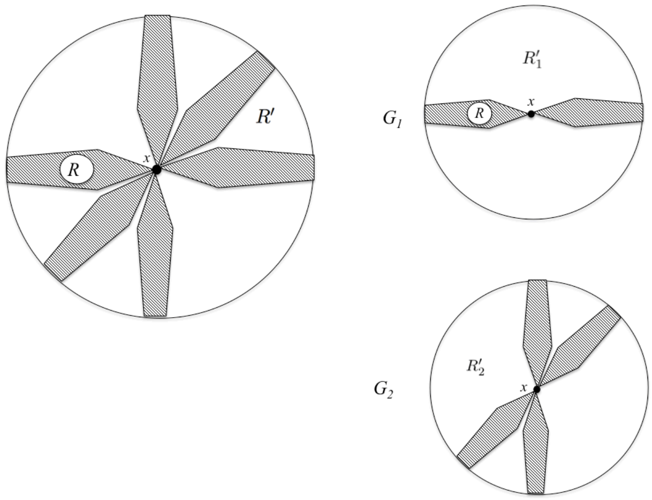


Figure 4.11: The structure of G, G_1 and G_2 in Case 5.2 when $R \neq R'$.

edge-blocks $x_1 = a_1, B_1, b_1a_2, B_2, ..., b_{k-1}a_k, B_k, b_k = x_2, C' \subseteq B_l$ for some $1 \leq l \leq k$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$. For each $1 \leq i \leq k$, we let F_i be an outer walk of B_i. By Theorem 3.1(a), B_l has a C'-Tutte trail T_l' from a_l to b_l containing e. For each $1 \leq i \leq k$, $i \neq l$, B_i has, by Theorem 3.1(a), an F_i-Tutte trail T_i' from a_i to b_i. Let $T'' = T_1' \cup T_2' \cup ... \cup T_k' \cup \{b_1a_2, b_2a_3, ..., b_{k-1}a_k\}$. By identifying x_1 and x_2, T becomes the desired C-Tutte closed trail T'' of G_1. Then $T'' \cup T_2$ is the desired C-Tutte closed trail of G. ■

4.2 Tutte trails of projective plane graphs

We start this section by proving the following lemma.

Lemma 4.5 *Let G be a 2-connected plane graph with outer walk C, and $x, y, z \in V(G)$. Then there exists a C-Tutte trail T from x to y such that either $z \in V(T)$, or $z \notin V(T)$ and the component of $G \backslash V(T)$ containing z is vertex-disjoint from C.*

Proof. If $z \in V(C)$, then by Theorem 3.1(a), G has a C-Tutte trail from x to y containing z. Hence we may assume that $z \notin V(C)$. We will consider in two cases.
Case 1: G has a 2-separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{u_1, u_2\}$, $E(G_i) \cap E(C) \neq \emptyset$ for $i = 1, 2$, $x, y \in V(G_1)$, and $z \in V(G_2)$.
Let G_i^* be the graph obtained from G_i by adding an edge $e_i = u_1u_2$ to the outer face of G_i such that $C \cap G_i \subset F_{G_i^*}$ for $i = 1, 2$. (See Fig. 4.12.) Note that G_1^* and G_2^* are 2-connected. By Theorem 3.1(a), G_1^* has an $F_{G_1^*}$-Tutte trail T_1 from x to y containing e_1, and G_2^* has an $F_{G_2^*}$-Tutte trail T_2 from z to z containing e_2. Hence $(T_1 - e_1) \cup (T_2 - e_2)$ is the desired C-Tutte trail of G from x to y containing z.

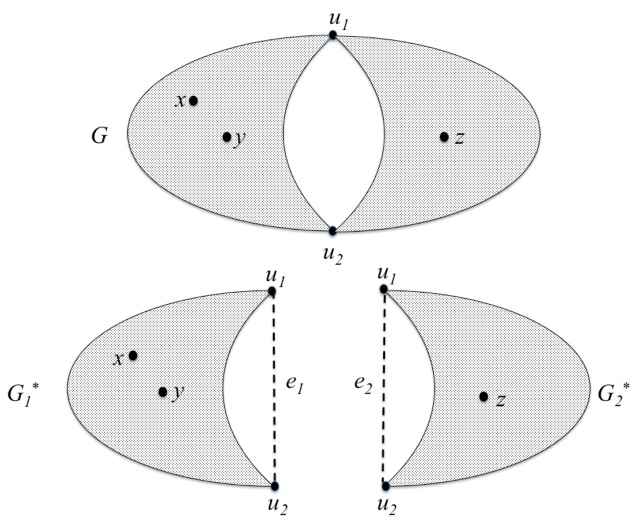


Figure 4.12: The structure of G, G_1^* and G_2^* in Case 1.

Case 2: There is no 2-separation of G as in Case 1.
By Theorem 3.1(a), G has a C-Tutte trail T from x to y. Let L be a component of

$G\backslash V(T)$ containing a vertex of C. Then L has exactly two edges connecting it to T. Let $B_1 = G\backslash V(L)$, and B_2 be the T-bridge of G containing L. Then (B_1, B_2) is a 2-separation of G such that $E(B_i) \cap E(C) \neq \emptyset$ for $i = 1, 2$, and $x, y \in V(B_1)$. Since there is no 2-separation of G as in Case 1, $z \notin V(L)$. This implies that either $z \in V(T)$, or the component of $G\backslash V(T)$ containing z is vertex-disjoint from C. Then T is the desired C-Tutte trail of G from x to y. ■

Kawarabashi and Ozeki [15] showed the following lemma.

Lemma 4.6 *Let G be a 2-connected graph embedded on the projective plane with representativity exactly 2, R be a face of G, C be the facial cycle of R, and $x, w \in V(C)$ such that $xw \in E(C)$. Suppose that G has (x, R)-width exactly 2. Then G can be decomposed into three plane graphs G_0, B and D such that G_0 is a 2-connected graph, $B = G_{S_1}$ where S_1 is a chain of blocks $b_0, B_1, b_1, B_2, ..., b_{n-1}, B_n, b_n$, and $D = G_{S_2}$ where S_2 is a chain of blocks $d_0, D_1, d_1, D_2, ..., d_{m-1}, D_m, d_m$ satisfying the following properties (G1 - G5). (Possibly, $|V(B)| = 1$ and/or $|V(D)| = 1$, in this case we take $b_0 = b_n$ and/or $d_0 = d_m$.)*

(G1) G_0 has four distinct vertices u_1, v_1, u_2 and v_2 such that they appear in F_{G_0} in this clockwise order.

(G2) G is obtained from G_0, B and D by identifying u_1 and b_0, u_2 and b_n, v_1 and d_0, and v_2 and d_m, respectively.

*(G3) $E(C) = E(F_{G_0}[u_1, v_1]) \cup E(F_{G_0}[u_2, v_2]) \cup \bigcup_{i=1}^{n} E(F_{B_i}[b_{i-1}, b_i]) \cup$
$\bigcup_{i=1}^{m} E(F_{D_i}[d_{i-1}, d_i])$.*

(G4) $x = d_k$ for some $0 \leq k \leq m-1$, and if $k \geq 1$, then $w \in V(F_{D_k}[d_{k-1}, d_k])$; otherwise, $w \in V(F_{G_0}[u_1, v_1])$.

(G5) If $x = d_0$, then G_0 has no 2-separation (K_1, K_2) such that $V(K_1 \cap K_2) = \{v_1, z\}$ and $z \in V(F_{G_0}[u_2, v_2])$. (See Fig. 4.13.)

We use Lemma 3.5, 3.6, 4.5, 4.6 and the ideas from [15] to prove the following theorem.

Theorem 4.7 *Let G be a 2-connected graph embedded on the projective plane, R be a face of G and C be the facial walk of R, $x \in V(C)$, and $y \in V(G)\backslash\{x\}$. Suppose $w \in V(C)$ with $wx \in E(C)$. Then there exist*

(i) a C-flap H in G with attachments a, b, c such that $y \notin V(H)\backslash\{a, b, c\}$ and

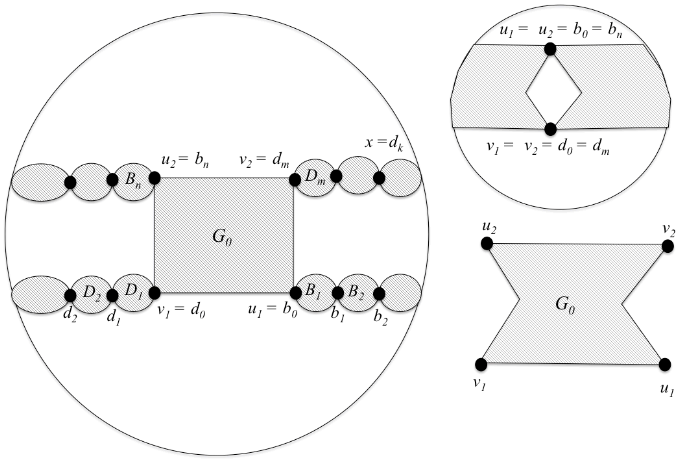


Figure 4.13: The structure of G in Lemma 4.6 when both B and D are not trivial (left), and when both B and D are trivial (right).

$x \in (V(H)\backslash\{a\}) \cup \{b\}$, and if H is non-trivial, then $w \in V(H)\backslash\{b\}$ and a, w, x, b appear in $C \cap H$ in this order;

(ii) a $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail T in $G\backslash(H\backslash\{a, b, c\})$ from b to y such that $a, c \in V(T)$.

(iii) Moreover, every component of $G\backslash V(T)$ which contains an essential cycle is vertex-disjoint from C.

Proof. If the representativity of G is zero, then G can be considered as a plane graph, and, by Theorem 3.1(a), G has a C-Tutte trail T from x to y. Then $H = \{x\}$ is the desired C-flap and T is the desired C-Tutte trail of G. Hence we may assume that the representativity of G is at least one. We prove the theorem by using induction on $|V(G)|$. If $|V(G)| = 3$, then we take $H = \{x\}$ and a path T from x to y of length two is the desired C-Tutte trail of G. So we assume $|V(G)| \geq 4$ and proceed to the induction step.

Case 1: The representativity of G is one.

Let R' be a face of G such that the R'-width of G is one. We let C' be the facial walk of R'. Let ϕ be be an essential closed curve passing through R' and intersecting G only at $v \in V(C')$. Then G can be redrawn as a plane graph G^* such that $C' = C_1 \cup C_2$ where C_1 is the outer cycle of G^*, $x, w \in V(C_1)$, C_2 is another facial cycle of G^*, and $C_1 \cap C_2 \neq \emptyset$.

Case 1.1: $R' = R$.

Note that $C = C' = C_1 \cup C_2$. (See Fig. 4.14.)

By Lemma 3.6, there exist: a C_1-flap H in G^* with attachments a, b, c such that $x \in$

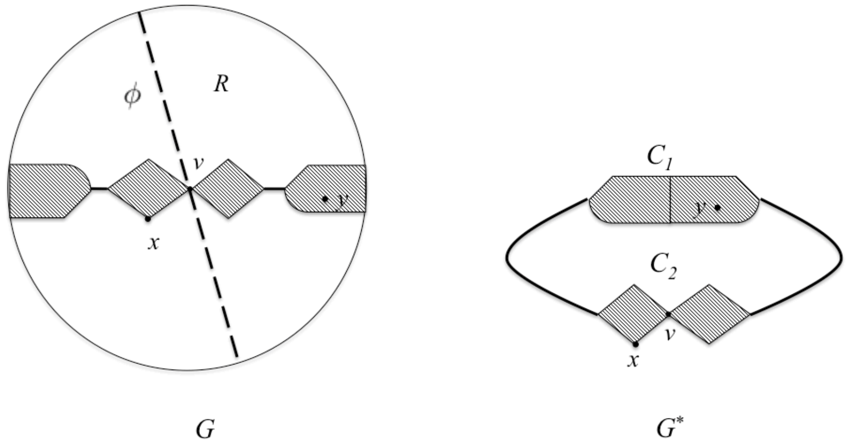


Figure 4.14: The structure of G and G^* in Case 1.1.

$(V(H)\backslash\{a\}) \cup \{b\}$, and if H is non-trivial, then $w \in V(H)\backslash\{b\}$ and a, w, x, b appear in $C_1 \cap H$ in this order; a $((C_1 \cup C_2)\backslash(H\backslash\{a, b, c\}))$-Tutte trail T in $G^*\backslash(H\backslash\{a, b, c\})$ from b to y such that $a, c \in V(T)$, and$T$ contains at least one vertex in $C_1 \cap C_2$. Furthermore, if $x \in V(C_1) \cap V(C_2)$, then $H = \{x\}$. Since every essential cycle of G contains every vertex of $C_1 \cap C_2$, H is the desired C-flap of G and T is the desired $(C\backslash(H\backslash\{a, b, c\})$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$.

Case 1.2: $R' \neq R$.

Then G^* can be redrawn as a plane graph K such that C is the outer cycle of K. We will define an C-Tutte trail T of K from x to y such that either $v \in V(T)$ or the component of $K\backslash V(T)$ containing v is vertex-disjoint from C.

Subcase 1.2.1: K has a 2-separation (K_1, K_2) such that $V(K_1 \cap K_2) = \{u_1, u_2\}$, $E(K_i) \cap E(C) \neq \emptyset$ for $i = 1, 2$, $x, y \in V(K_1)$, and $v \in V(K_2)$.

Let K_i^* be the graph obtained from K_i by adding an edge $e_i = u_1u_2$ to the outer face of K_i such that $C \cap K_i \subset F_{K_i^*}$ for $i = 1, 2$.(See Fig. 4.15.) Note that K_1^* and K_2^* are 2-connected. By Theorem 3.1(a), K_1^* has an $F_{K_1^*}$-Tutte trail T_1 from x to y containing e_1, and K_2^* has an $F_{K_2^*}$-Tutte trail T_2 from v to v containing e_2. Hence

$H = \{x\}$ is the desired C-flap of G, and $(T_1 - e_1) \cup (T_2 - e_2)$ is the desired C-Tutte trail of G from x to y.

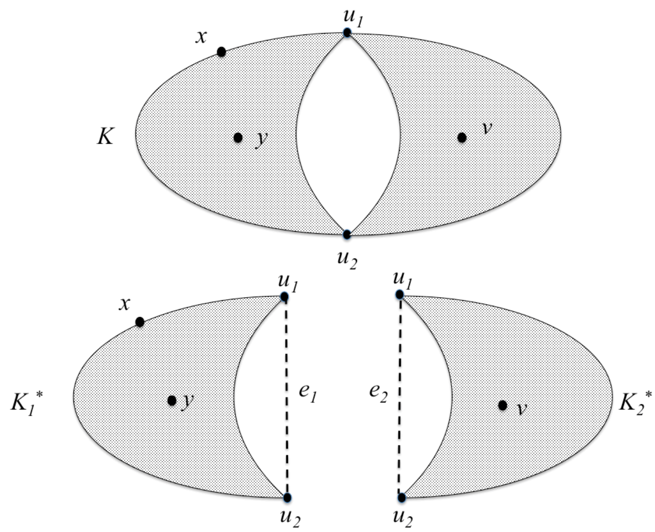


Figure 4.15: The structure of K, K_1^* and K_2^* in Subcase 1.2.1.

Subcase 1.2.2: There is no 2-separation of K as in Subcase 1.2.1.
By Theorem 3.1(a), K has a C-Tutte trail T from x to y. Let L be a component of $K \backslash V(T)$ containing a vertex of C. Then L has exactly two edges connecting it to T. Let $B_1 = K \backslash V(L)$, and B_2 be the T-bridge of K containing L. Then (B_1, B_2) is a 2-separation of K such that $E(B_i) \cap E(C) \neq \emptyset$ for $i = 1, 2$, and $x, y \in V(B_1)$. Since there is no 2-separation of K as in Subcase 1.2.1, $v \notin V(L)$. Hence $H = \{x\}$ is the desired C-flap of G, and T is the desired C-Tutte trail of G from x to y.

Case 2: The representativity and the (x, R)-width of G are exactly two.
By Lemma 4.6, G can be decomposed into three plane graphs G_0, B and D such that G_0 is a 2-connected plane graph, $B = G_{S_1}$ where S_1 is a chain of blocks $b_0, B_1, b_1, B_2, ..., b_{n-1}, B_n, b_n$, and $D = G_{S_2}$ where S_2 is a chain of blocks d_0, D_1, d_1, D_2, ..., d_{m-1}, D_m, d_m satisfying properties (G1 - G5). (See Fig. 4.13.) Note that, from (G4), $x = d_k$ for some $0 \leq k \leq m-1$, and if $k \geq 1$, then $w \in V(F_{D_k}[d_{k-1}, d_k])$; otherwise, $w \in V(F_{G_0}[u_1, v_1])$.

For each $1 \leq i \leq m$, by Theorem 3.1(a), D_i has an F_{D_i}-Tutte trail T_{D_i} from d_{i-1} to d_i. Let $T(d_i, d_j) = \bigcup_{p=i+1}^{j} T_{D_p}$ when $0 \leq i < j \leq m$. We can construct T_{B_i} and

$T(b_i, b_j)$ for $0 \leq i < j \leq n$. We will consider three subcases as follows.

Case 2.1: $y \in V(G_0)$.

By Lemma 3.5(A1), there exists a $F_{G_0}[u_1, v_2]$-Tutte subgraph in G_0 consisting of two edge-disjoint trails T_1 and T_2 such that T_1 and T_2 connect $\{v_2, y\}$ and $\{u_1, u_2\}$.

Suppose $v_1 \notin V(T_1 \cup T_2)$. Then the component L of $G_0 \backslash V(T_1 \cup T_2)$ containing v_1 has two edges connecting it to $T_1 \cup T_2$. Let L^* be the $(T_1 \cup T_2)$-bridge of G_0 containing L with attachment $a \in F_{G_0}[u_1, v_1]$ and $c \in F_{G_0}[v_1, u_2]$. Then $H = L^* \cup D$ is the desired C-flap of G with attachment a, v_2, c, and $T = T_1 \cup T_2 \cup T(b_0, b_n)$ is the desired $(C \backslash (H \backslash \{a, v_2, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, v_2, c\})$. (See Fig. 4.16.) Note that since every essential cycle of G must contain either u_1 or v_2 and both u_1 and v_2 are in $V(T)$, then no component of $G \backslash V(T)$ can contain an essential cycle.

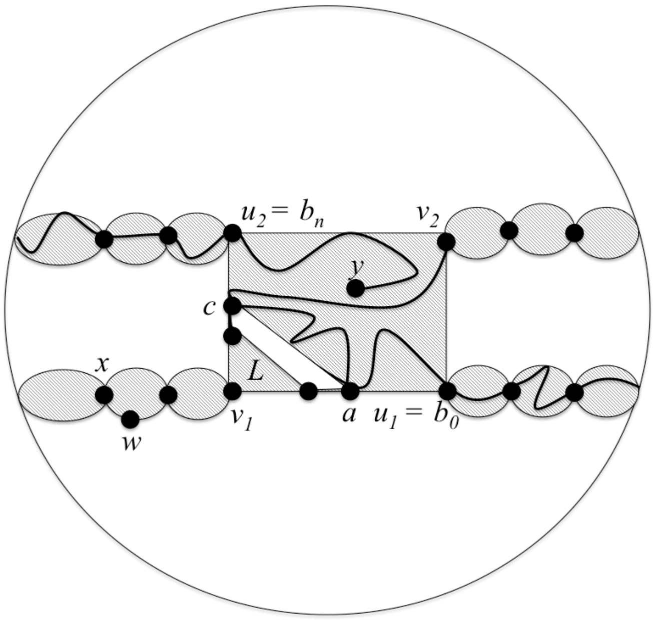


Figure 4.16: The structure of G in Case 2.1 when $v_1 \notin V(T_1 \cup T_2)$.

Suppose $v_1 \in V(T_1 \cup T_2)$. Let $D^* = \bigcup_{j=1}^{k} D_j$. We will define T_x and T_{d_0} on D^* as follows.

- If $k = 0$, then $x = d_0$ and let $T_x = T_{d_0} = \{d_0\}$.
- When D^* is 2-edge-connected, we let $T_{d_0} = \{d_0\}$. By Theorem 3.1(a), we can choose T_x to be an F_{D^*}-Tutte trail of D^* from x to x containing d_0.

- When D^* is not 2-edge-connected, we let Q_x (respectively, Q_{d_0}) be an edge-block of D^* containing x(respectively, d_0). Let z_x(respectively, z_{d_0}) be the vertex of $Q_x\backslash\{x\}$ (respectively, $Q_{d_0}\backslash\{d_0\}$) such that z_x(respectively, z_{d_0}) has a neighbor in $D^*\backslash Q_x$(respectively, $D^*\backslash Q_{d_0}$). By Theorem 3.1(a), Q_x has an F_{Q_x}-Tutte trail T_x from x to x containing z_x, and Q_{d_0} has an $F_{Q_{d_0}}$-Tutte trail T_{d_0} from d_0 to d_0 containing z_{d_0}. (If $Q_x = \{x\}$ and/or $Q_{d_0} = \{d_0\}$, we let $T_x = \{x\}$ and/or $T_{d_0} = \{d_0\}$.)

Then $H = \{x\}$ is the desired C-flap of G, and $T = T_1 \cup T_2 \cup T_x \cup T_{d_0} \cup T(x, d_m) \cup T(b_0, b_n)$ is the desired C-Tutte trail of G. (See Fig.4.17.) Note that since every essential cycle of G must contain either u_1 or v_2, but both u_1 and v_2 are in $V(T)$, then a component of $G\backslash V(T)$ does not contain an essential cycle.

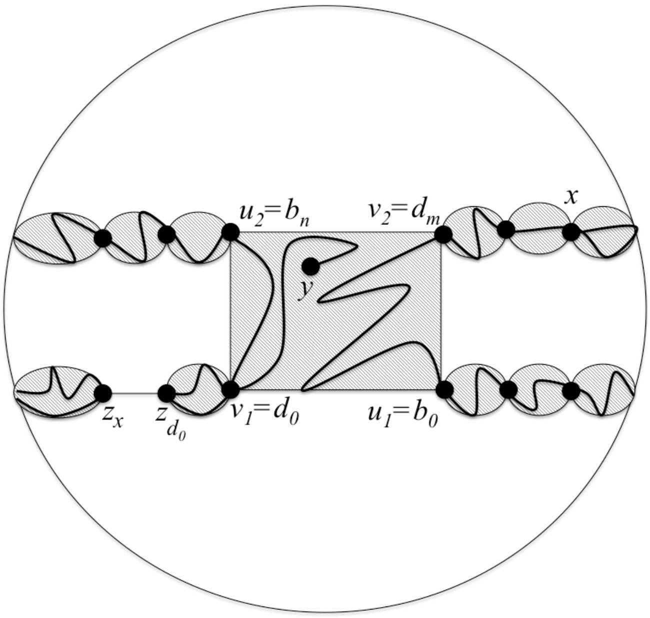


Figure 4.17: The structure of G in Case 2.1 when $v_1 \in V(T_1 \cup T_2)$.

Case 2.2: $y \in V(B)\backslash\{u_1, u_2\}$.
By Theorem 3.1(a), G_0 has an F_{G_0}-Tutte trail T from v_2 to u_2 containing u_1. Let $y \in V(B_l)$ for some $1 \le l \le n$. By Theorem 3.1(a), B_l has an F_{B_l}-Tutte trail T'_{B_l} from y to b_l containing b_{l-1}. Let $B^* = \bigcup_{i=1}^{l-1} B_i$. We will define T_{b_0} and $T_{b_{l-1}}$ on B^* as follows.

- If $l = 1$, then $b_{l-1} = b_0$ and let $T_{b_0} = T_{b_{l-1}} = \{b_0\}$.

- When B^* is 2-edge-connected, we let $T_{b_{l-1}} = \{b_{l-1}\}$. By Theorem 3.1(a), we can choose T_{b_0} to be an F_{B^*}-Tutte trail of B^* from b_0 to b_0 containing b_{l-1}.

- When B^* is not 2-edge-connected, we let Q_{b_0} (respectively, $Q_{b_{l-1}}$) be an edge-block of B^* containing b_0(respectively, b_{l-1}). Let z_{b_0}(respectively, $z_{b_{l-1}}$) be the vertex of $Q_{b_0}\backslash\{b_0\}$ (respectively, $Q_{b_{l-1}}\backslash\{b_{l-1}\}$) such that z_{b_0}(respectively, $z_{b_{l-1}}$) has a neighbor in $B^*\backslash Q_{b_0}$(respectively, $B^*\backslash Q_{b_{l-1}}$). By Theorem 3.1(a), Q_{b_0} has an $F_{Q_{b_0}}$-Tutte trail T_{b_0} from b_0 to b_0 containing z_{b_0}, and $Q_{b_{l-1}}$ has an $F_{Q_{b_{l-1}}}$-Tutte trail $T_{b_{l-1}}$ from b_{l-1} to b_{l-1} containing $z_{b_{l-1}}$. (If $Q_{b_0} = \{b_0\}$ and/or $Q_{b_{l-1}} = \{b_{l-1}\}$, we let $T_{b_0} = \{b_0\}$ and/or $T_{b_{l-1}} = \{b_{l-1}\}$.)

Suppose $v_1 \notin V(T)$. Then the component L of $G_0\backslash V(T)$ containing v_1 has two edges connecting it to T. Let L^* be the T-bridge of G_0 containing L with attachment $a \in F_{G_0}[u_1, v_1]$ and $c \in F_{G_0}[v_1, u_2]$. Then $H = L^* \cup D$ is the desired C-flap of G with attachment a, v_2, c, and $T^* = T \cup T'_{B_l} \cup T_{b_0} \cup T_{b_{l-1}} \cup T(b_l, b_n)$ is the desired $(C\backslash(H\backslash\{a, v_2, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, v_2, c\})$. (See Fig.4.18.) Note that since every essential cycle of G must contain either u_1 or v_2 and both u_1 and v_2 are in $V(T^*)$, then no component of $G\backslash V(T^*)$ can contain an essential cycle.

Suppose $v_1 \in V(T)$. Then we define T_x and T_{d_0} in the same way as in Case 2.1. Then $H = \{x\}$ is the desired C-flap of G, and $T^* = T \cup T_x \cup T_{d_0} \cup T(x, d_m) \cup T'_{B_l} \cup T_{b_0} \cup T_{b_{l-1}} \cup T(b_l, b_n)$ is the desired C-Tutte trail of G. (See Fig.4.19.) Note that since every essential cycle of G must contain either u_1 or v_2, but both u_1 and v_2 are in $V(T^*)$, then a component of $G\backslash V(T^*)$ does not contain an essential cycle.

Case 2.3: $y \in V(D)\backslash\{v_1, v_2\}$.
Let $y \in V(D_l)$ for some $1 \le l \le m$. In this case, we will show that H is trivial so we can ignore the vertex w. Assume without the loss of generality that $l \le k$. By Theorem 3.1(a), D_l has an F_{D_l}-Tutte trail T'_{D_l} from y to d_{l-1} containing d_l. Let $D' = \bigcup_{j=l+1}^{k} D_j$. We will define T_x and T_{d_l} on D' as follows.

- If $k = l + 1$, then $x = d_l$ and let $T_x = T_{d_l} = \{d_l\}$.

- When D' is 2-edge-connected, we let $T_{d_l} = \{d_l\}$. By Theorem 3.1(a), we can choose T_x to be an $F_{D'}$-Tutte trail of D' from x to x containing d_l.

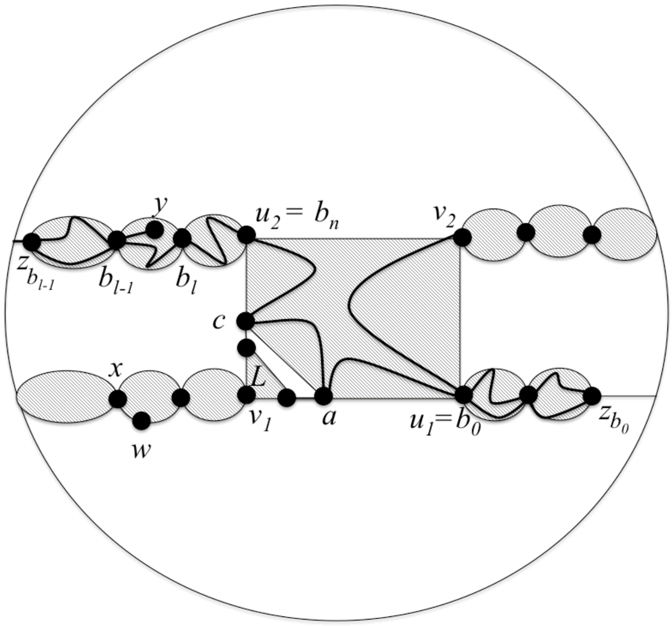


Figure 4.18: The structure of G in Case 2.2 when $v_1 \notin V(T)$.

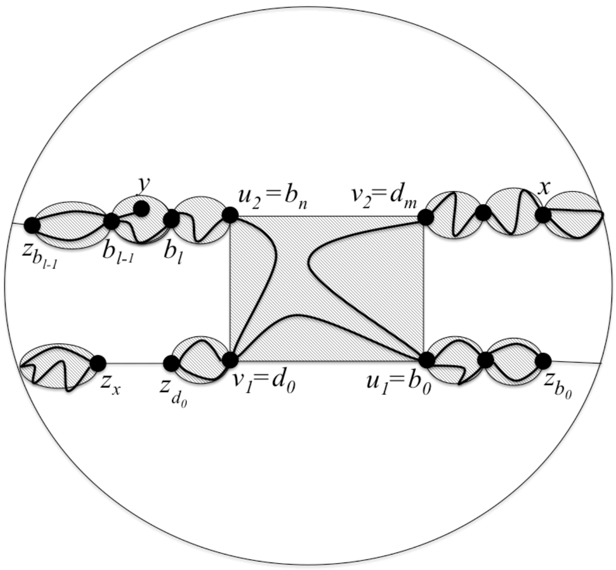


Figure 4.19: The structure of G in Case 2.2 when $v_1 \in V(T)$.

- When D' is not 2-edge-connected, we let Q_x (respectively, Q_{d_l}) be an edge-block of D' containing x(respectively, d_l). Let z_x (respectively, z_{d_l}) be the vertex of $Q_x \backslash \{x\}$ (respectively, $Q_{d_l} \backslash \{d_l\}$) such that z_x (respectively, z_{d_l}) has a neighbor in $D' \backslash Q_x$ (respectively, $D' \backslash Q_{d_l}$). By Theorem 3.1(a), Q_x has an F_{Q_x}-Tutte trail T_x from x to x containing z_x, and Q_{d_l} has an $F_{Q_{d_l}}$-Tutte trail T_{d_l} from d_l to d_l containing z_{d_l}. (If $Q_x = \{x\}$ and/or $Q_{d_l} = \{d_l\}$, we let $T_x = \{x\}$ and/or $T_{d_l} = \{d_l\}$.)

By Lemma 3.5, there exists a $F_{G_0}[u_1, v_2]$-Tutte subgraph in G_0 consisting of two edge-disjoint trails T_1 and T_2 such that T_1 and T_2 connect $\{v_1, v_2\}$ and $\{u_1, u_2\}$. Then $H = \{x\}$ is the desired C-flap of G, and $T^* = T_1 \cup T_2 \cup T_x \cup T_{d_l} \cup T(x, d_m) \cup T'_{D_l} \cup T(d_0, d_{l-1}) \cup T(b_0, b_n)$ is the desired C-Tutte trail of G. (See Fig.4.20.) Note that since every essential cycle of G must contain either u_1 or v_2, but both u_1 and v_2 are in $V(T^*)$, then a component of $G \backslash V(T^*)$ does not contain an essential cycle.

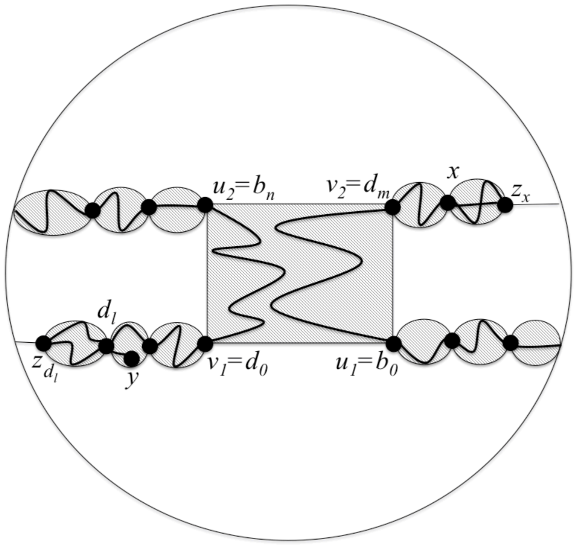


Figure 4.20: The structure of G in Case 2.3.

Case 3: The representativity of G is at least two, the (x, R)-width is at least three, and G has a 2-separation (G_1, G_2) such that $x \in V(G_1 \cap G_2)$ and $E(C) \cap E(G_i) \neq \emptyset$ for $i = 1, 2$.

By Lemma 4.2, we can choose a 2-separation (G_1, G_2) of G such that G_1 contains an essential cycle, and G_2 does not contain an essential cycle. Suppose that $V(G_1) \cap V(G_2) = \{x, z\}$.

Let G_1^* be a new graph obtained from G_1 by adding a new edge e_1 from x to z such that e_1 is in R. Note that G_1^* is 2-connected. Let $w^* = w$ if $w \in V(G_1)$; otherwise $w^* = z$. Let R^* be the face of G_1^* with the facial walk $C^* = (C \cap G_1) \cup \{e_1\}$. (See Fig. 4.21.)

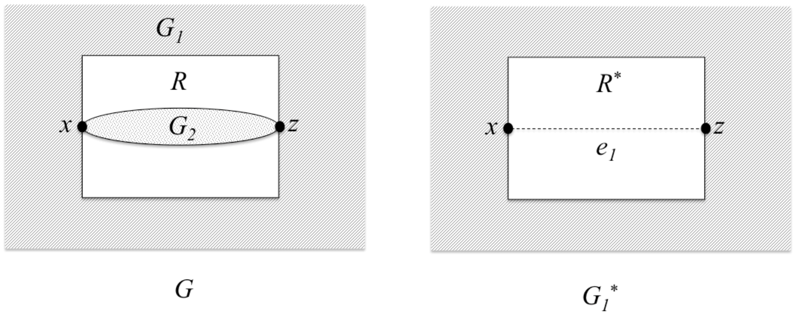


Figure 4.21: The structure of G and G_1^* in Case 3.

Case 3.1: $y \in V(G_1)$.
By induction, there exist: a C^*-flap H^* in G_1^* with attachments a, b, c such that $y \notin V(H) \backslash \{a, b, c\}$ and $x \in (V(H^*) \backslash \{a\}) \cup \{b\}$, and if H^* is non-trivial, then $w^* \in V(H^*) \backslash \{b\}$ and a, w^*, x, b appear in $C^* \cap H^*$ in this order; a $(C^* \backslash (H^* \backslash \{a, b, c\}))$-Tutte trail T^* in $G_1^* \backslash (H^* \backslash \{a, b, c\})$ from b to y such that $a, c \in V(T^*)$. Moreover, every component of $G_1^* \backslash V(T^*)$ which contains an essential cycle is vertex-disjoint from C^*.

Subcase 3.1.1: $e_1 \in V(T^*)$.
Note that if $w = w^* \in V(G_1)$, then H^* is either trivial or non-trivial with $x = b$; and if $z = w^*$, then H^* is trivial. Let G_2^* be a new graph obtained from G_2 by adding a new edge e_2 from x to z such that $E(F_{G_2} \cap C) \subseteq E(F_{G_2^*})$. Since G_2^* is a 2-connected plane graph, then, by Theorem 3.1(a), G_2^* has an $F_{G_2^*}$-Tutte trail T_2 from x to x containing e_2. (See Fig. 4.22.)

Then $H = H^*$ is the desired C-flap, and $(T^* - e_1) \cup (T_2 - e_2)$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$.

Subcase 3.1.2: $e_1 \in E(H^*)$.
Note that H^* is non-trivial. Let $H = (H^* - e_1) \cup G_2$. Then H is the desired C-flap and T^* is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig.4.23.)

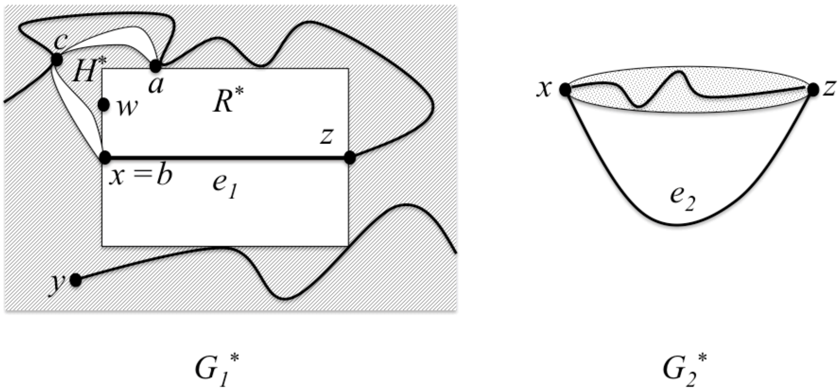


Figure 4.22: The structure of G_1^* and G_2^* in Case 3.1.1 when H^* is non-trivial.

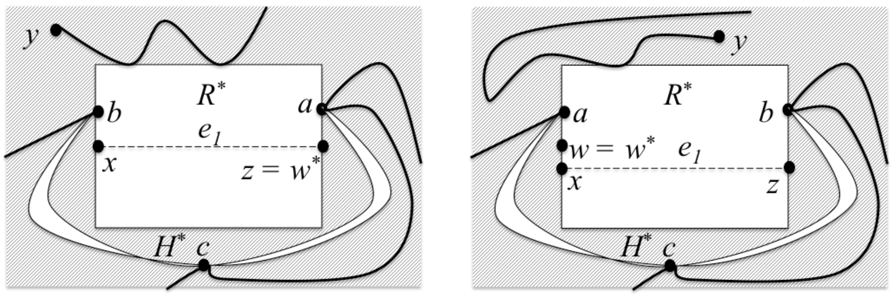


Figure 4.23: The structure of G_1^* in Case 3.1.2 when $z = w^*$ (left side) and $w = w^*$ (right side).

Subcase 3.1.3: $e_1 \notin E(T^*) \cup E(H^*)$ and $z \in V(T^*)$.
Note that $w^* = w \in V(G_1)$ so H^* is either trivial or non-trivial with $x = b$. We will define trails T_x and T_z of G_2 as follows.

- When G_2 is 2-edge-connected, we let $T_z = \{z\}$. By Theorem 3.1(a), we can choose T_x to be an F_{G_2}-Tutte trail of G_2 from x to x containing z.

- When G_2 is not 2-edge-connected, we let L_x (respectively, L_z) be an edge-block of G_2 containing x(respectively, z). Let v_x(respectively, v_z) be the vertex of $L_x \backslash \{x\}$ (respectively, $L_z \backslash z$) such that v_x(respectively, v_z) has a neighbor in $G_2 \backslash L_x$ (respectively, $G_2 \backslash L_z$). By Theorem 3.1(a), L_x has an F_{L_x}-Tutte trail T_x from x to x containing v_x, and L_z has an F_{L_z}-Tutte trail T_z from z to

z containing v_z. (If $L_x = \{x\}$ and/or $L_z = \{z\}$, we let $T_x = \{x\}$ and/or $T_z = \{z\}$.)

Then $H = H^*$ is the desired C-flap, and $T = T^* \cup T_x \cup T_z$ is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig.4.24.) Note that if G_2 is not 2-edge-connected, then the component $G_2\backslash(L_x \cup L_z)$ has exactly two edges connecting it to T.

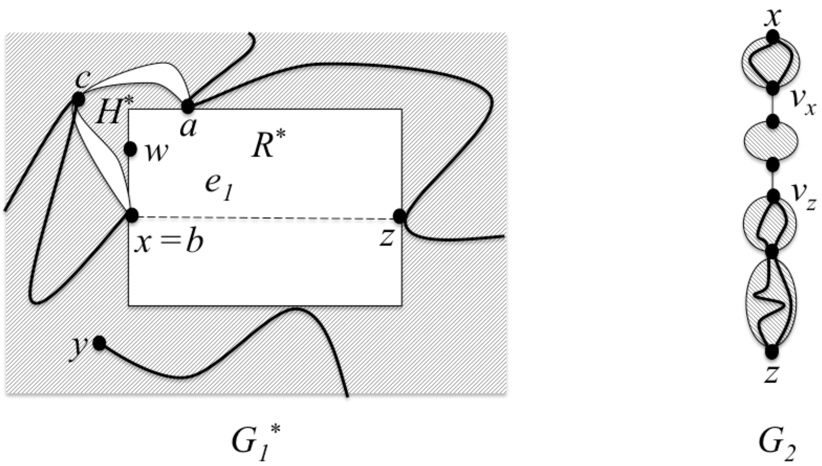


Figure 4.24: The structure of G_1^* and G_2 in Case 3.1.3 when H^* is non-trivial

Subcase 3.1.4: $e_1 \notin E(T^*) \cup E(H^*)$ and $z \notin V(T^*)$.
Note that $w^* = w \in V(G_1)$ so H^* is either trivial or non-trivial with $x = b$. Since $z \notin V(T^*)$, then the component B of $G_1^*\backslash V(T^*)$ containing z has two edges e_1, e_B connecting it to $V(T^*)$. We let $K = G_2 \cup B$ and z^* be the end vertex of e_B in B. (See Fig.4.25.) We will define a trail T_x of K as follows.

- When K is 2-edge-connected, by Theorem 3.1(a), we can choose T_x to be an F_K-Tutte trail of K from x to x containing z^*.

- When K is not 2-edge-connected, we let L_x be an edge-block of K containing x. Let v_x be the vertex of $L_x\backslash\{x\}$ such that v_x has a neighbor in $K\backslash L_x$. By Theorem 3.1(a), L_x has an F_{L_x}-Tutte trail T_x from x to x containing v_x. (If $L_x = \{x\}$, we let $T_x = \{x\}$.)

Then $H = H^*$ is the desired C-flap, and $T = T^* \cup T_x$ is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.25.) Note that if K is not 2-edge-connected, then the component $K\backslash L_x$ has exactly two edges connecting it to T.

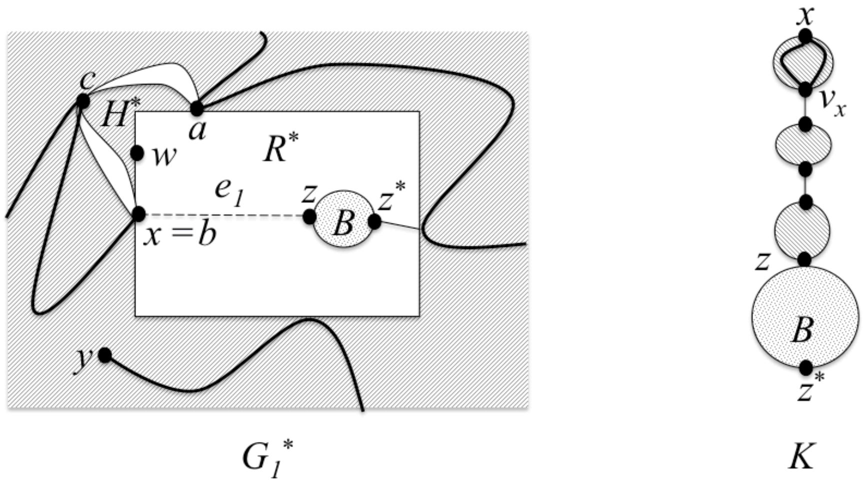


Figure 4.25: The structure of G_1^* and K in Case 3.1.4 when H^* is non-trivial.

Case 3.2: $y \in V(G_2)\backslash V(G_1)$.
By Theorem 4.4, there is a C^*-Tutte closed trail T_1 of G_1^* containing e_1 such that every component of $G_1^*\backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C. (See Fig.4.26.)

Let $G_2 = H_S$ where S is a chain of edge-blocks $z = a_1, B_1, b_1a_2, B_2, ..., b_{k-1}a_k, B_k, b_k = x$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$. Suppose $y \in V(B_m)$ for some $1 \leq m \leq k$. By Theorem 3.1(a), B_i has an F_{B_i}-Tutte trail T_{B_i} from a_i to b_i for each $1 \leq i \leq m-1$, B_m has an F_{B_m}-Tutte trail T_{B_m} from a_m to y containing b_m, and B_k has an F_{B_k}-Tutte trail T_x from x to x containing a_k. (If $m = k$, we let $T_x = \{x\}$.)

Hence $H = \{x\}$ is the desired C-flap, and $(T_1 - e_1) \cup T_x \cup \{b_1a_2, b_2a_3, ..., b_{m-1}a_m\} \cup \bigcup_{i=1}^{m} T_{B_i}$ is the desired C-Tutte trail in G. (See Fig. 4.26.)

Case 4: The representativity of G is at least two, the (x, R)-width is at least three, and G has a 2-separation (G_1, G_2) such that $x \in V(G_1 \cap G_2)$ and $E(C) \subseteq E(G_1)$. By Lemma 4.2, we can choose a 2-separation (G_1, G_2) of G such that G_1 contains an

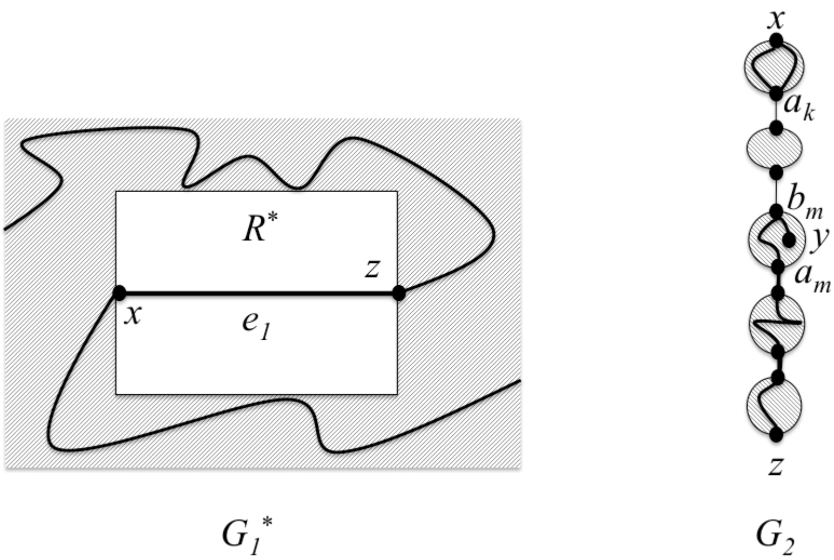


Figure 4.26: The structure of G_1^* and G_2 in Case 3.2.

essential cycle, G_2 does not contain an essential cycle, $V(G_1) \cap V(G_2) = \{x, z\}$, and $|V(G_1)|$ is minimal among all such 2-separation (K_1, K_2) of G with $x \in V(K_1 \cap K_2)$ and $C \subseteq K_1$.

Suppose that $V(G_1) \cap V(G_2) = \{x, z\}$ and $|V(G_1)|$ is minimal among all such 2-separation (K_1, K_2) of G with $x \in V(K_1 \cap K_2)$ and $C \subseteq K_1$.

Since G is 2-connected with representativity at least two, by Lemma 4.2, exactly one of G_1 and G_2 does not contain an essential cycle.

Let R_1 (respectively, R_2) be the face of G_1 (respectively, G_2) with facial walk C_1 (respectively, C_2) such that R_1 contains $G_2 \backslash \{x, z\}$(respectively, R_2 contains $G_1 \backslash \{x, z\}$) and $x, z \in V(C_1)$ (respectively, $x, z \in V(C_2)$). (See Fig. 4.27.)

Case 4.1: G_1 contains an essential cycle.

Note that G_2 is a plane graph and $F_{G_2} = C_2$. Let G_1'(respectively, G_1'') be a new graph obtained from G_1 by adding a new edge e_1 (respectively, two new edges e_1, f_1) from x to z such that both edges are in R_1. (See Fig. 4.28.) Let $y^* = y$ if $y \in V(G_1)$; otherwise $y^* = z$. Let $G_1^* \in \{G_1', G_1'', G_1\}$.

By induction, there exist: a C-flap H^* in G_1^* with attachments a, b, c such that $y^* \notin V(H^*) \backslash \{a, b, c\}$ and $x \in (V(H^*) \backslash \{a\}) \cup \{b\}$, and if H^* is non-trivial, then $w \in V(H^*) \backslash \{b\}$ and a, w, x, b appear in $C \cap H^*$ in this order; a $(C \backslash (H^* \backslash \{a, b, c\}))$-

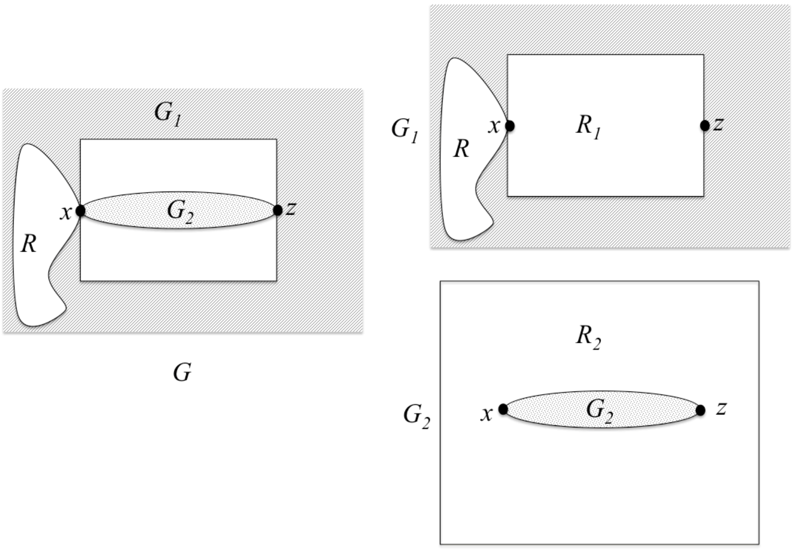


Figure 4.27: The structure of G, G_1 and G_2 in Case 4 when G_1 contains an essential cycle.

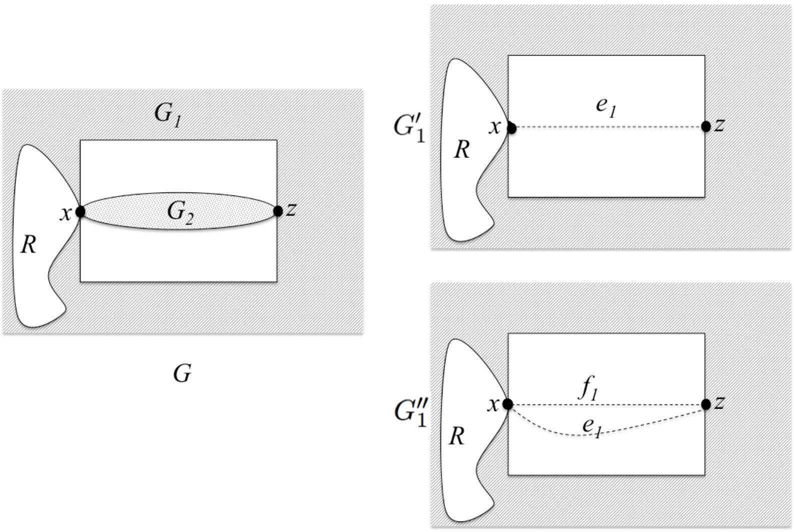


Figure 4.28: The structure of G, G_1' and G_1'' in Case 4.1.

Tutte trail T^* in $G_1^*\backslash(H^*\backslash\{a,b,c\})$ from b to y^* such that $a, c \in V(T^*)$. Moreover, every component of $G_1^*\backslash V(T^*)$ which contains an essential cycle is vertex-disjoint from C^*.

Case 4.1.1: $y \in V(G_1)$ and G_2 is 2-edge-connected.

Note that $y^* = y$. We let $G_1^* = G_1''$

Subcase 4.1.1(a): $e_1 \in E(T^*)$ but $f_1 \notin E(T^*)$

Note that H^* is either trivial or non-trivial with $x = b$. Note that if H^* is non-trivial, then either $f_1 \notin E(H^*)$, or $f_1 \in E(H^*)$ and $z = c$. Since G_2 is a 2-edge-connected plane graph, by Theorem 3.1(a), G_2 has an C_2-Tutte trail T_2' from x to z. Then $H = H^*$ is the desired C-flap, and $(T^* - e_1) \cup T_2'$ is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.29.)

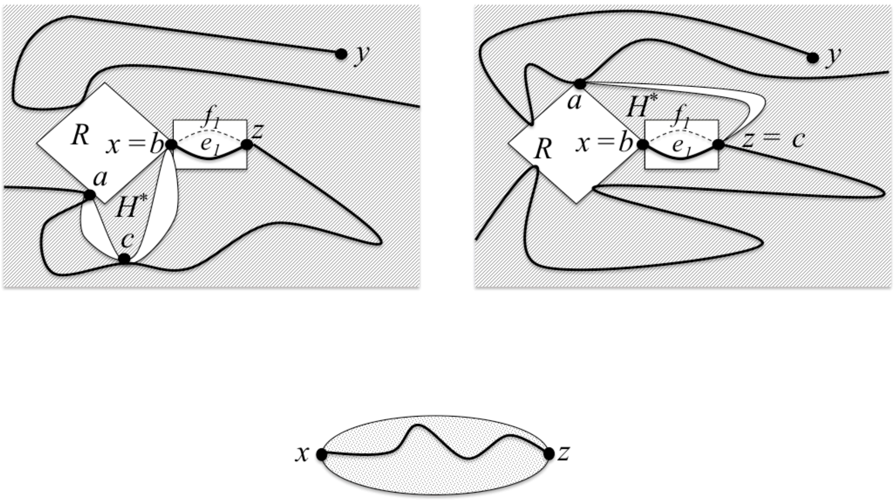


Figure 4.29: The structure of G_1^* when $z \notin V(H^*)$ (left), G_1^* when $z = c \in V(H^*)$ (right), and G_2 (bottom) in Case 4.1.1(a).

Subcase 4.1.1(b): Both $e_1, f_1 \in E(T^*)$.

Note that H^* is either trivial or non-trivial with $x = b$. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2 from x to x containing z. Then $H = H^*$ is the desired C-flap, and $(T^* - e_1 - f_1) \cup T_2$ is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.30.)

Subcase 4.1.1(c): $e_1 \in E(H^*)$ and $f_1 \notin E(T^*)$.

Note that H^* is non-trivial. Then $H = (H^* - e_1 - f_1) \cup G_2$ is the desired C-flap and T^* is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.31.)

Subcase 4.1.1(d): $e_1, f_1 \notin E(T^*) \cup E(H^*)$.

Note that H^* is either trivial or non-trivial with $x = b$. Suppose $z \in V(T^*) \cup V(H^*)$. Note that if $z \in V(H^*)$, then $z = c \in V(T^*)$. By Theorem 3.1(a), G_2 has a C_2-Tutte

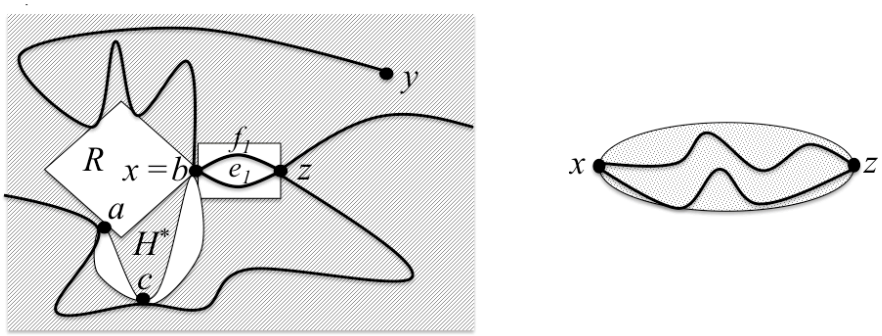


Figure 4.30: The structure of G_1^* and G_2 in Case 4.1.1(b) when H^* is non-trivial.

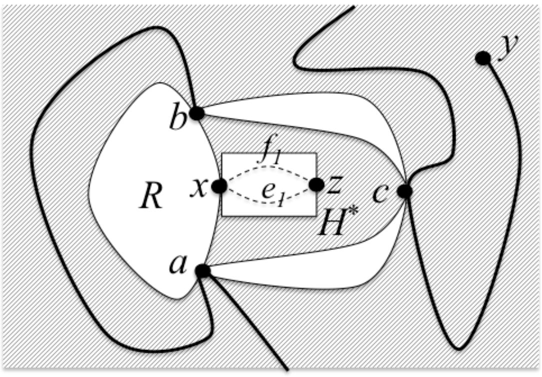


Figure 4.31: The structure of G_1^* in Case 4.1.1(c) when $e_1, f_1 \in E(H^*)$.

trail T_2 from x to x containing z. Then $H = H^*$ is the desired C-flap of G, and $T^* \cup T_2$ is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.32.)

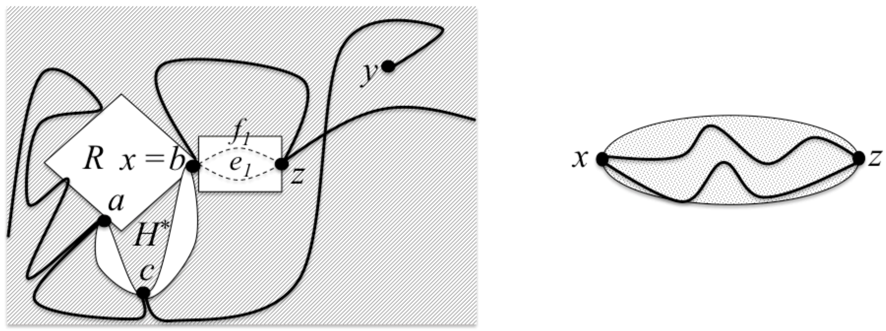


Figure 4.32: The structure of G_1^* and G_2 in Case 4.1.1(d) when $z \in V(T^*)$.

Suppose $z \notin V(T^*) \cup V(H^*)$. Then the component D of $G_1^* \backslash V(T^*)$ has at most three edges e_1, f_1, h connecting it to $V(T^*)$, and $u \in G_1^* \backslash V(D)$ is the end vertex of h. (Possibly, D contains an essential cycle.) This is a contradiction since $(G_1 \backslash D, G_2 \cup D)$ is a 2-separation of G with $x, u \in V(G_1 \backslash D) \cup V(G_2 \cup D)$, and $V(G_1 \backslash D \subset V(G_1)$. (See Fig. 4.33.)

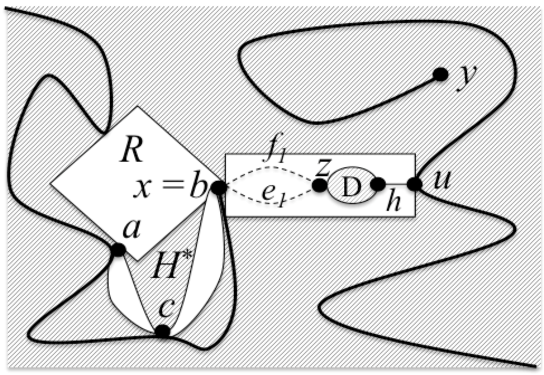


Figure 4.33: The structure of G_1^* in Case 4.1.1(d) when $z \notin V(T^*)$.

Case 4.1.2: $y \in V(G_1)$ and G_2 is not 2-edge-connected.
Note that $y^* = y$. We let $G_1^* = G_1'$

Subcase 4.1.2(a): $e_1 \in E(T^*)$. Note that H^* is either trivial or non-trivial with $x = b$. Let G_2' be a new graph obtained from G_2 by adding a new edge e_2 from x to z such that $e_2 \in R_2$. Since G_2' is a 2-edge-connected plane graph, by Theorem 3.1(a), G_2' has an $F_{G_2'}$-Tutte trail T_2 from x to x containing e_2. Then $H = H^*$ is the desired C-flap, and $(T^* - e_1) \cup (T_2 - e_2)$ is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.34.)

Subcase 4.1.2(b): $e_1 \in E(H^*)$.
Note that H^* is non-trivial. Then $H = (H^* - e_1) \cup G_2$ is the desired C-flap and T^* is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.35.)

Subcase 4.1.2(c): $e_1 \notin E(T^*) \cup E(H^*)$.
Note that H^* is either trivial or non-trivial with $x = b$. Let L_x (respectively, L_z) be an edge-block of G_2 containing x(respectively, z). Note that $L_x \neq L_z$ since G_2 is not 2-edge-connected. Let v_x(respectively, v_z) be the vertex of $L_x\backslash\{x\}$ (respectively, $L_z\backslash z$) such that v_x(respectively, v_z) has a neighbor in $G_2\backslash L_x$ (respectively, $G_2\backslash L_z$). By Theorem 3.1(a), L_x has an F_{L_x}-Tutte trail T_x from x to x containing v_x, and L_z

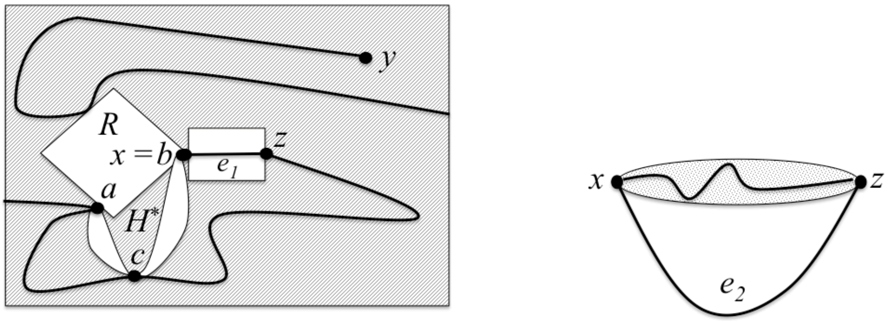


Figure 4.34: The structure of G_1^* and G_2' in Case 4.1.2(a) when H^* is non-trivial.

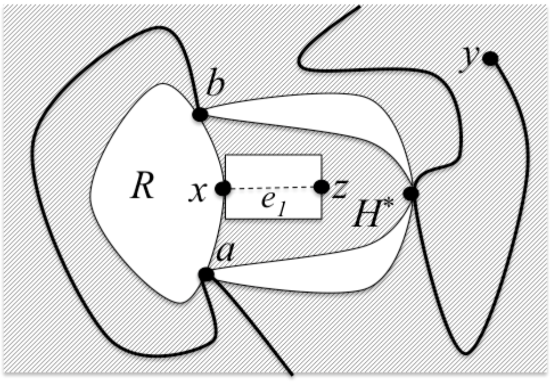


Figure 4.35: The structure of G_1^* in Case 4.1.2(b).

has an F_{L_z}-Tutte trail T_z from z to z containing v_z. (If $L_x = \{x\}$ and/or $L_z = \{z\}$, we let $T_x = \{x\}$ and/or $T_z = \{z\}$.)

Suppose $z \in V(T^*) \cup V(H^*)$. Then $H = H^*$ is the desired C-flap of G, and $T^* \cup T_x \cup T_z$ is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.36.)

Suppose $z \notin V(T^*) \cup V(H^*)$. Let D be the component D of $G_1^*\backslash V(T^*)$ which contains z. (Possibly, D contains an essential cycle.) If D has exactly two edges e_1, h connecting it to $V(T^*)$, then we contradict the 2-connectivity of G_1.

Hence D has three edges connecting it to $V(T^*)$. Then $H = H^*$ is the desired C-flap of G, and $T^* \cup T_x$ is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. Note that the component $(G_2\backslash L_x) \cup D$ has three edges connecting it to T. (See Fig. 4.38.)

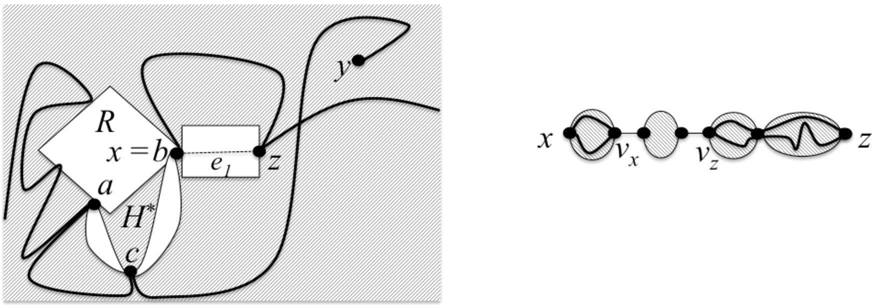


Figure 4.36: The structure of G_1^* in Case 4.1.2(c) when $z \in V(T^*)$, and G_2.

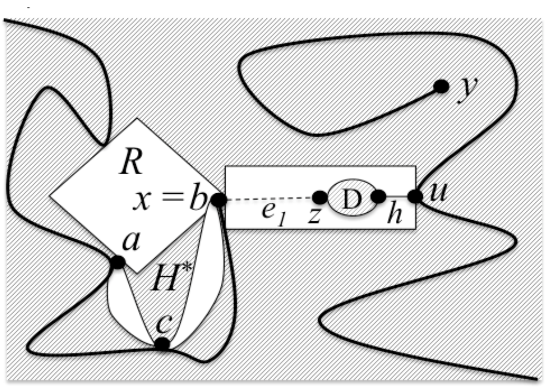


Figure 4.37: The structure of G_1^* in Case 4.1.2(c) when $z \notin V(T^*)$ and D has exactly two edges connecting it to $V(T^*)$.

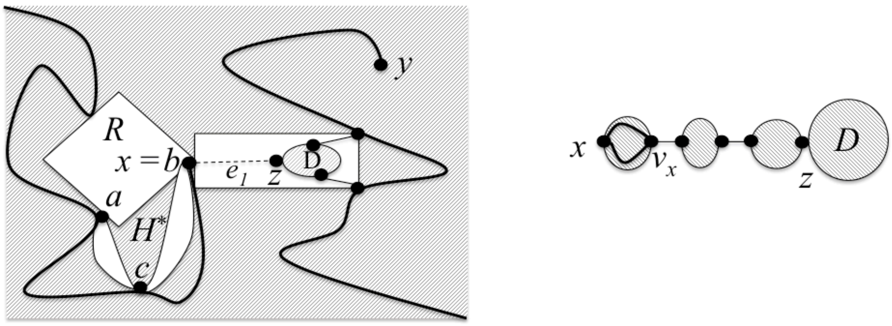


Figure 4.38: The structure of G_1^* when $z \notin V(T^*)$ and D has three edges connecting it to $V(T^*)$, and $G_2 \cup D$ in Case 4.1.2(c).

Case 4.1.3: $y \in V(G_2) \backslash V(G_1)$.

We let $G_1^* = G_1$ and $y^* = z$. Note that G_1 is 2-connected and $x, z \in V(C_1)$ where C_1 is the facial walk of R_1.

Let $G_2 = H_S$ where S is a chain of edge-blocks $z = a_1, B_1, b_1a_2, B_2, ..., b_{k-1}a_k, B_k, b_k = x$ and $a_i, b_i \in V(B_i)$ for all $1 \le i \le k$. Suppose $y \in V(B_m)$ for some $1 \le m \le k$. By Theorem 3.1(a), B_i has an F_{B_i}-Tutte trail T_{B_i} from a_i to b_i for each $1 \le i \le m-1$, B_m has an F_{B_m}-Tutte trail T_{B_m} from a_m to y containing b_m, and B_k has an F_{B_k}-Tutte trail T_x from x to x containing a_k. (If $m = k$, we let $T_x = \{x\}$.) Let $T_y = \{b_1a_2, b_2a_3, ..., b_{m-1}a_m\} \cup \bigcup_{i=1}^m T_{B_i}$

Subcase 4.1.3(a): Either H^* is trivial, or H^* is non-trivial with $x = b$.
Then $H = H^*$ is the desired C-flap of G, and $T^* \cup T_x \cup T_y$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig. 4.39.)

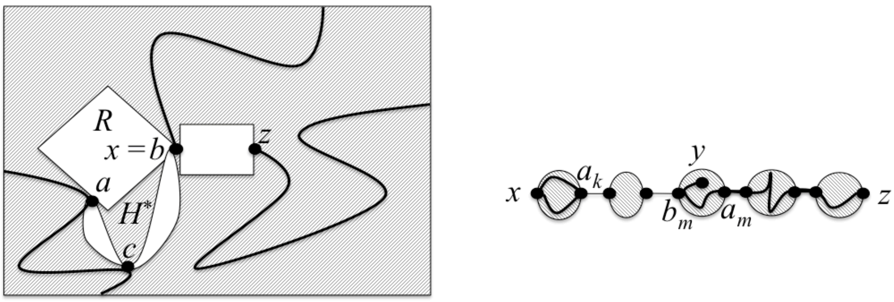


Figure 4.39: The structure of G_1^* and G_2 in Case 4.1.3(a).

Subcase 4.1.3(b): H^* is non-trivial with $x \neq b$, and $b \in V(C_1)$.
Note that x and z are in the facial walk of R_1 in $G_1^* = G_1$ and $z \notin H^* \backslash \{a, b, c\}$. (Possibly, $z \in \{b, c\}$ in this case.) Let P be the path of $H^* \cap C$ from x to b. Let $M = H_U$ where U is the chain of edge-blocks in the edge-block tree from x to b. Then $U = a_1, A_1, b_1a_2, A_2, ..., b_{n-1}a_n, A_n, b_n = b$, where $a_1 = x, b_n = b$, $V(A_i) \cap P \neq \emptyset$, and $a_i, b_i \in V(A_i)$ for all $1 \le i \le n$. By Theorem 3.1(a), A_i has an F_{A_i}-Tutte trail T_{A_i} from a_i to b_i for all $1 \le i \le n$.

Then $H = H^* \backslash (H_S \backslash \{x\})$ is the desired C-flap of G with attachment a, x, c, and $T^* \cup T_x \cup T_y \cup \bigcup_{i=1}^n T_{A_i} \cup \{b_1a_2, b_2a_3, ..., b_{n-1}a_n\}$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig. 4.40.)

Subcase 4.1.3(c): H^* is non-trivial with $x \neq b$, and $a \in V(C_1)$.
Note that x and z are in the facial walk of R_1 in $G_1^* = G_1$ and $z \notin H^* \backslash \{a, b, c\}$.

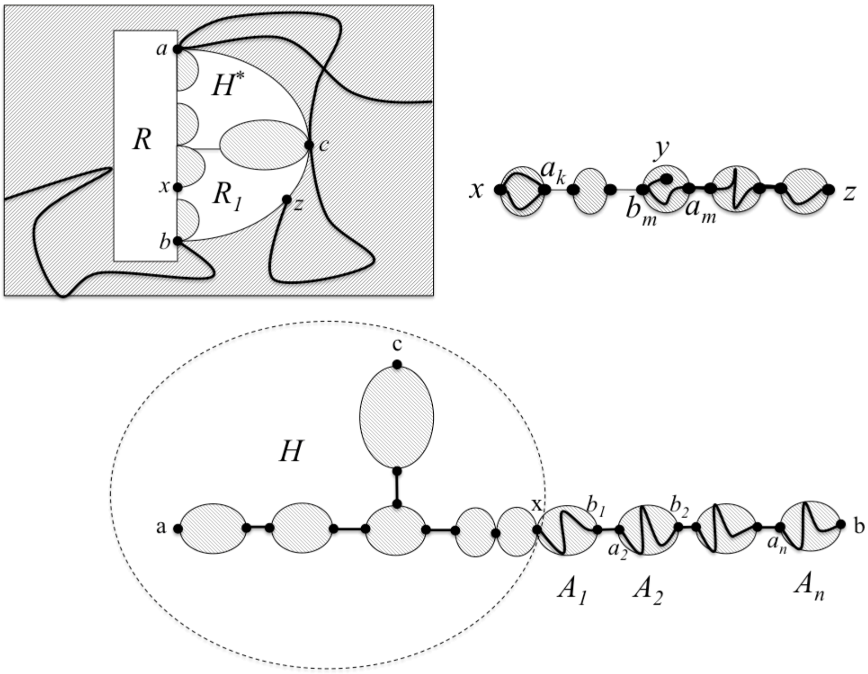


Figure 4.40: The structure of G (left), H^* (bottom) and G_2 (right) in Case 4.1.3(b)

(Possibly, $z \in \{a, c\}$ in this case.) Let P be the path of $H^* \cap C$ from x to b. Note that $a, c \notin V(P)$. By Lemma 3.3, there exists a P-Tutte subgraph T of H consisting of three edge-disjoint trails T_H, T_a and T_c such that T_H is a trail from x to b, T_a and T_c are closed trails containing a and c, respectively, $x \notin V(T_a) \cup V(T_c)$, and there is at most one component of $H \backslash V(T)$ which contains a vertex of F_H and has three edges connecting it to T. Note that $H^* \cap C$ is the maximal subwalk of F_{H^*} from a to b which contains P and $c \notin V(H^* \cap C)$.

Since every path of H^* from a to c must intersect x, H does not have an edge-block K such that $a, \{x, b\}, c$ belong to different components of $H \backslash K$. Then, by Lemma 3.3, T is an $(H^* \cap C)$-Tutte subgraph of H^*.

Then $H = \{x\}$ is the desired C-flap of G, and $T^* \cup T_x \cup T_y \cup T$ is the desired C-Tutte trail in G. (See Fig. 4.41.)

Case 4.2: G_1 does not contain an essential cycle.
Note that G_2 contains an essential cycle. Let $G_2 = H_S$ where S is a chain of edge-blocks $x = a_1, B_1, b_1 a_2, B_2, \ldots, b_{k-1} a_k, B_k, b_k = z$ and $a_i, b_i \in V(B_i)$ for all $1 \leq i \leq k$.

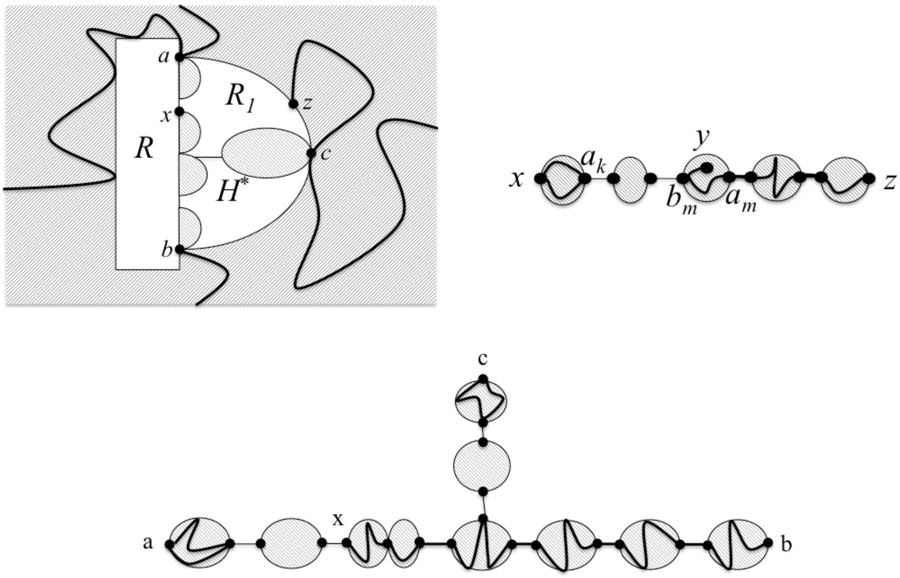


Figure 4.41: The structure of G (left), H^* (bottom) and G_2 (right) in Case 4.1.3(c)

Let G_2^* be a new graph obtained from G_2 by adding a new edge e_2 from x to z in the face R_2. Let R_2^* be a face of G_2^* with the facial walk C_2^* such that $R_2^* \subset R_2$. (See Fig. 4.42.)

Case 4.2.1: $y \in V(G_1)$.
By Theorem 4.4, there is a C_2^*-Tutte closed trail T_2 of G_2^* containing e_2 such that every component of $G_2^* \backslash V(T_2)$ which contains an essential cycle is vertex-disjoint from C_2.

By Theorem 3.1(a), G_1 has a C-Tutte trail T_1 from z to y containing x. Then $H = \{x\}$ is the desired C-flap of G, and $T_1 \cup (T_2 - e_2)$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig. 4.43.)

Case 4.2.2: $y \in V(G_2) \backslash V(G_1)$.
Let $w^* \in E(C_2^*) \backslash \{z\}$. By induction, there exist: a C_2^*-flap H^* in G_2^* with attachments a, b, c such that $y \notin V(H^*) \backslash \{a, b, c\}$ and $x \in (V(H^*) \backslash \{a\}) \cup \{b\}$, and if H^* is non-trivial, then $w^* \in V(H^*) \backslash \{b\}$ and a, w^*, x, b appear in $C_2^* \cap H^*$ in this order; a $(C_2^* \backslash (H^* \backslash \{a, b, c\}))$-Tutte trail T^* in $G_2^* \backslash (H^* \backslash \{a, b, c\})$ from b to y such that $a, c \in V(T^*)$. Moreover, every component of $G_2^* \backslash V(T^*)$ which contains an essential cycle is vertex-disjoint from C_2^*.

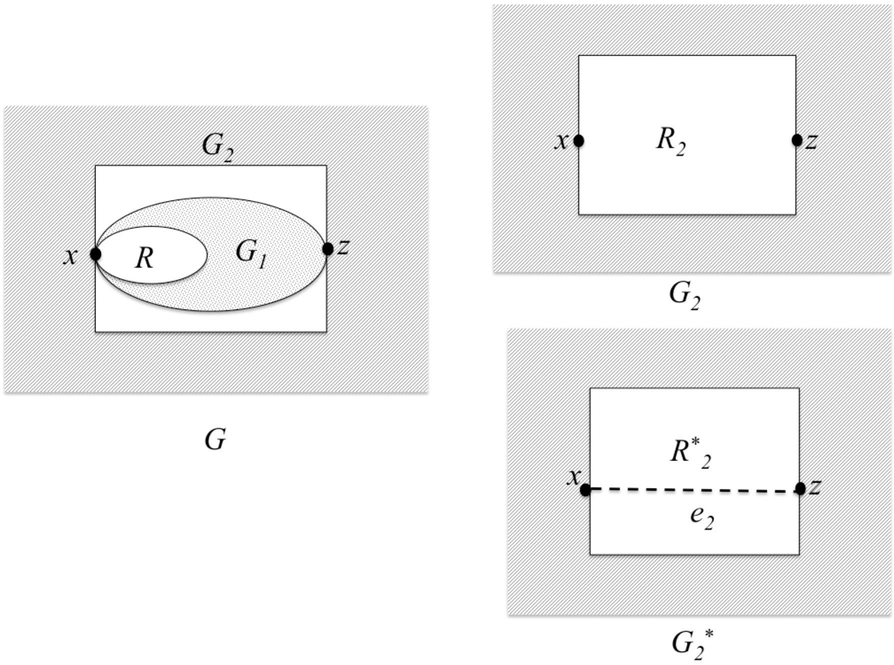


Figure 4.42: The structure of G, G_2 and G_2^* in Case 4.2.

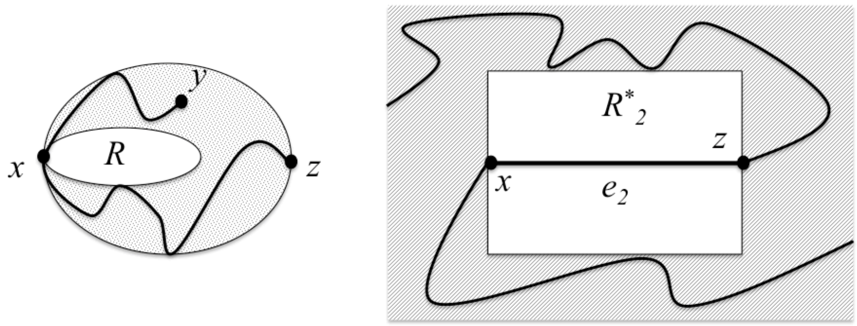


Figure 4.43: The structure of G_1 and G_2^* in Case 4.2.1.

Suppose P is the path of $C_2^* \cap H^*$ from x to b. By Lemma 3.3, there exists a P-Tutte subgraph of H^* consisting of three edge-disjoint trails T_{H^*}, T_a and T_c such that T_{H^*} is a trail from x to b, T_a and T_c are closed trails containing a and c, respectively, and $x \notin V(T_a) \cup VT_c)$. Let $T = T^* \cup T_{H^*} \cup T_a \cup T_c$.

Subcase 4.2.2(a): $e_2 \in E(T)$.

Note that either $e_2 \in E(T^*)$, or $e_2 \in E(T_{H^*})$. By Theorem 3.1(a), G_1 has a C-Tutte trail T_1 from x to z. Then $H = \{x\}$ is the desired C-flap of G, and $T \cup T_1$ is the desired C-Tutte trail in G. (See Fig. 4.44.)

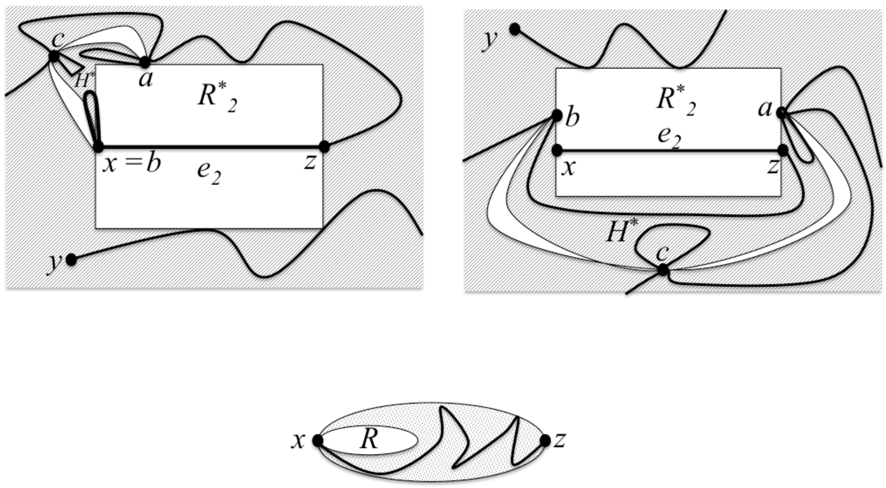


Figure 4.44: The structure of G_1 (bottom), G_2^* when $z \notin V(H^*)$(left), and G_2^* when $z \in V(H^*)$ (right) in Case 4.2.2(a).

Subcase 4.2.2(b): $e_2 \notin E(T)$.

Suppose $z \in V(T)$. Then $x = b$. By Theorem 3.1(a), G_1 has a C-Tutte trail T_1' from x to x containing z. Then $H = \{x\}$ is the desired C-flap of G, and $T \cup T_1'$ is the desired C-Tutte trail in G. (See Fig. 4.45.)

Suppose $z \notin V(T)$. Let D be the component of $G_2^* \backslash V(T^*)$ which contains z. Note that D has no essential cycle since it contains a vertex of C_2^*. Then D has exactly two edges e_2, f connecting it to $V(T^*)$. Let u be the end vertex of f in D. We will define a trail T_x of $G_1' = G_1 \cup D$ as follows.

- When G_1' is 2-edge-connected. By Theorem 3.1(a), we can choose T_x to be an $F_{G_1'}$-Tutte trail of G_1' from x to x containing u.

- When G_1' is not 2-edge-connected, we let L_x be an edge-block of G_1' containing x. Let v_x be the vertex of $L_x \backslash \{x\}$ such that v_x has a neighbor in $G_1' \backslash L_x$. By Theorem 3.1(a), L_x has an F_{L_x}-Tutte trail T_x from x to x containing v_x. (If $L_x = \{x\}$, we let $T_x = \{x\}$.)

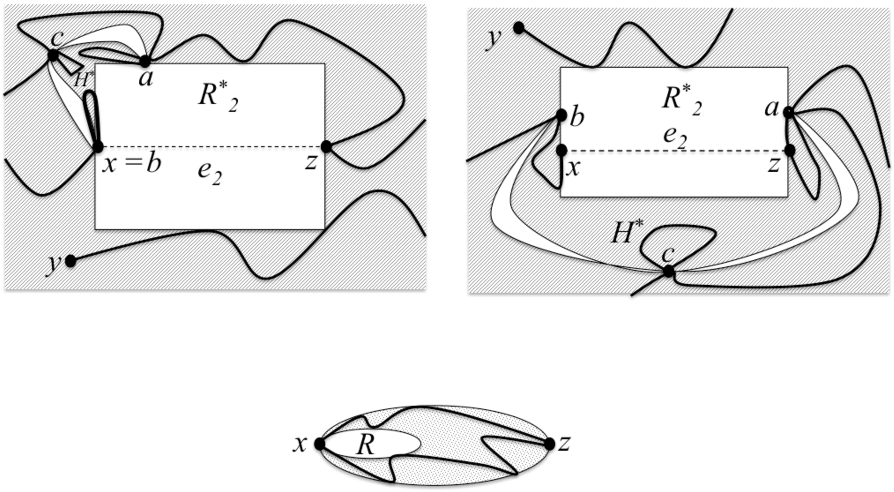


Figure 4.45: The structure of G_1 (bottom), G_2^* when $z \in V(T)\backslash V(H^*)$(left), and G_2^* when $z \in V(H) \cup V(H^*)$ (right) in Case 4.2.2(b).

Then $H = \{x\}$ is the desired C-flap of G, and $T \cup T_x$ is the desired C-Tutte trail in G. (See Fig. 4.46.)

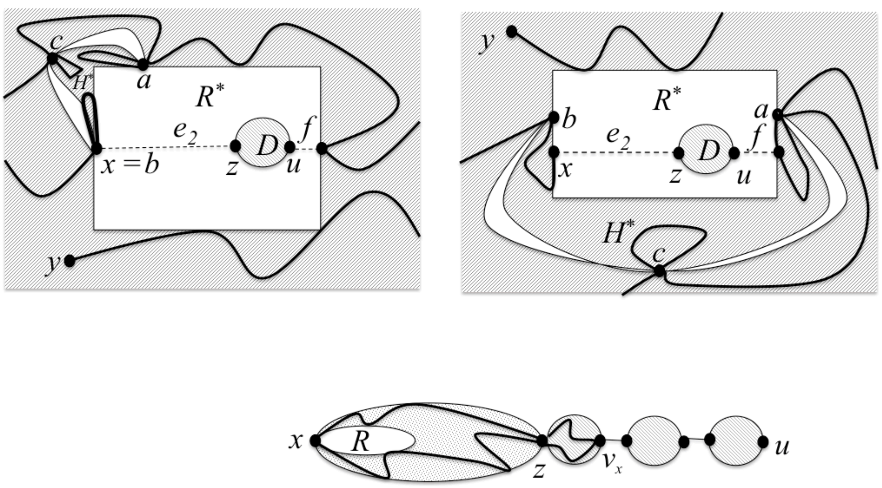


Figure 4.46: The structure of G_1' (bottom), G_2^* when $z \notin V(T) \cup V(H^*)$(left), and G_2^* when $z \in V(H^*)\backslash V(T)$ (right) in Case 4.2.2(b).

Case 5: The representativity of G is at least two, the (x, R)-width is at least three,

and $G\backslash\{x\}$ is 2-connected.
Let $G^* = G\backslash\{x\}$. Let z be the neighbor of x in C with $z \neq w$. Let R^* be the face of G^* containing R, and C^* be the facial walk of R^*. We let $w^* \in V(C^*)$ with $w^*z \in E(C^*)\backslash E(C)$. (See Fig. 4.47.)

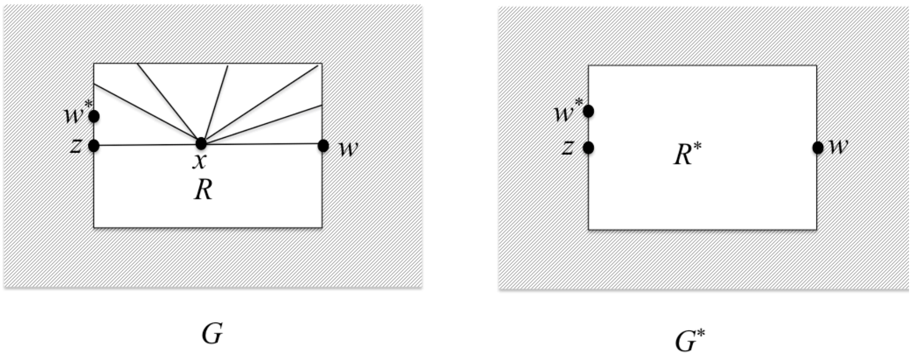


Figure 4.47: The structure of G and G^* in Case 5.

By induction, there exist; a C^*-flap H^* in G^* with attachments a^*, b^*, c^* such that $y \notin V(H^*)\backslash\{a^*, b^*, c^*\}$ and $z \in (V(H^*)\backslash\{a^*\}) \cup \{b^*\}$, and if H^* is non-trivial, then $w^* \in V(H^*)\backslash\{b^*\}$ and a^*, w^*, z, b^* appear in $C^* \cap H^*$ in this order; a $(C^*\backslash(H^*\backslash\{a^*, b^*, c^*\}))$-Tutte trail T^* in $G^*\backslash(H^*\backslash\{a^*, b^*, c^*\})$ from b^* to y such that $a^*, c^* \in V(T^*)$. Moreover, every component of $G^*\backslash V(T^*)$ which contains an essential cycle is vertex-disjoint from C^*.

Case 5.1: H^* is trivial.
Then $z = a^* = b^* = c^*$. If $w \notin V(T^*)$, we let K be the component of $G^*\backslash V(T^*)$ containing w. Note that K is a plane graph since $V(K) \cap V(C^*) \neq \emptyset$.

Subcase 5.1.1: $w \notin V(T^*)$ and $V(C^*)\backslash V(C) \subset V(K)$.
Note that z is a neighbor of K on $V(T^*)$. Let K' be the subgraph of G induced by $V(K) \cup \{x, z\}$ and $B_{K'}$ be the edge-block of K' containing w. Note that $x, z \in V(B_{K'})$. Let $d \in V(B_{K'})\backslash\{z\}$ such that d has a neighbor in $G\backslash B_{K'}$. By Theorem 3.1(a), $B_{K'}$ has an $F_{B_{K'}}$-Tutte trail $T_{K'}$ from x to z containing d. (See Fig. 4.48.) Note that the component of $K'\backslash B_{K'}$ has two edges connecting it to $T^* \cup T_{K'}$. Then $H = \{x\}$ is the desired C-flap of G and $T^* \cup T_{K'}$ is the desired C-Tutte trail of G.

Subcase 5.1.2: Either $w \in V(T^*)$, or $w \notin V(T^*)$ and $V(C^*)\backslash V(C) \not\subset V(K)$.
In this case, we will show that the desired C-flap H is either trivial, or non-trivial

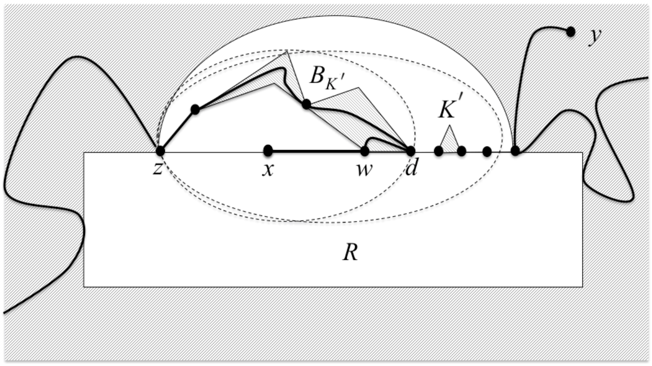


Figure 4.48: The structure of T^* and $T_{K'}$ in Case 5.1.1.

with attachments a, x, c for some vertices $a, c \in V(G)$. Let $J_1, J_2, ..., J_n$ be the distinct components of $G^*\backslash V(T^*)$ such that $J_i \neq K$ and J_i contains a vertex of $V(C^*)\backslash V(C)$ for all $1 \leq i \leq n$. (See Fig. 4.49.) Then J_i is a plane graph and has exactly two edges connecting it to T^*. If J_i has at most one edge connecting it to x for some $1 \leq i \leq n$, we let $T_{J_i} = \emptyset$.

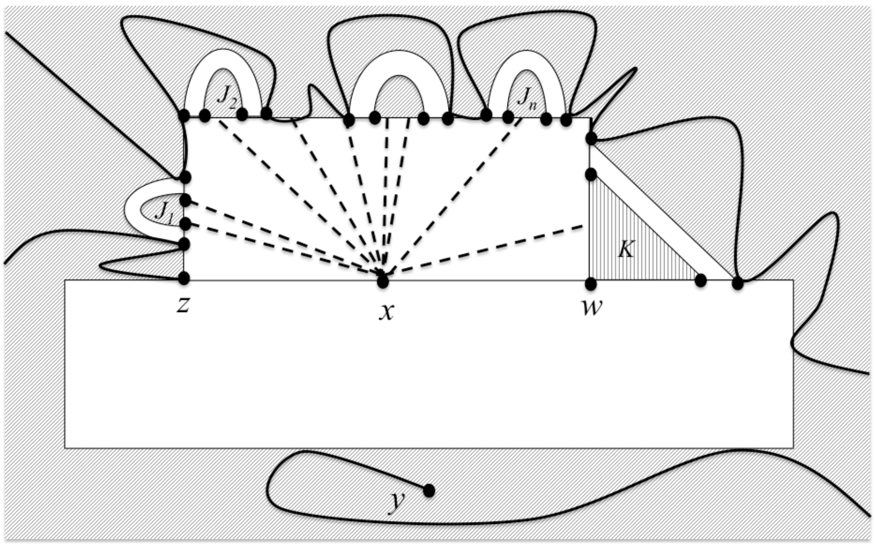


Figure 4.49: The structure of K and T^* in Case 5.1.2 when $w \notin V(T^*)$.

Next, suppose that J_i has at least two edges connecting it to x for some $1 \leq i \leq n$. Let J_i' be the subgraph of G induced by $V(J_i) \cup \{x\}$ and L be the edge-block of J_i' containing x. Let $x_1, x_2 \in V(J_i')$ such that x_i has a neighbor in $G\backslash V(J_i')$ for $i = 1, 2$. Then $J_i'\backslash L$ has two components L_1 and L_2 containing x_1 and x_2, respectively. (Possibly, $L_1 = \emptyset$ and/or $L_2 = \emptyset$.) Let $x_i' \in V(L)$, for $i = 1, 2$ such that x_i' has a neighbor in L_i. (If $L_i = \emptyset$, then we let $x_i = x_i'$.) By Theorem 3.1(a), L has an F_L-Tutte trail T_{J_i} from x to x containing x_1'. (See Fig. 4.50.) Note that if $x_2' \notin V(T_{J_i})$, then the component Q of $L\backslash V(T_{J_i})$ containing x_2' has at most two edges connecting it to T_{J_i} and so $L_2 \cup Q$ has at most three edges connecting it $T^* \cup T_{J_i}$.

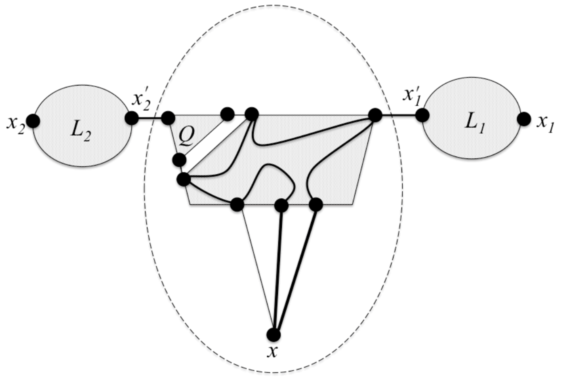


Figure 4.50: The structure of T_{J_i} in J_i' in Case 5.1.2.

If $w \in V(T^*)$, we let $H = \{x\}$, and if $w \notin V(T^*)$, we let H be the $(T^* \cup \{x\})$-bridge of G containing w. Note that H is a C-flap with three vertices of attachment, one of which is x. Let a and c be the other two vertices of attachment of H such that $a \in V(C)$ and $c \in V(C^*)\backslash V(C)$. Note that $V(H) = V(K) \cup \{a, b, c\}$. (See Fig. 4.51.)

Then H is the desired C-flap of G with attachments a, x, c and $T^* \cup \{xz\} \cup \bigcup_{i=1}^{n} T_{J_i}$ is the desired $(C\backslash(H\backslash\{a, x, c\}))$-Tutte trail of $G\backslash(H\backslash\{a, x, c\})$.

Case 5.2: H^* is non-trivial and $a^* \in V(C^*)\backslash V(C)$.

Note that H^* is a plane graph and either $b^* \in V(C)$ or $b^* \in V(C^*\backslash C)$. In this case, we will show that the desired C-flap H is either trivial, or non-trivial with attachments a, x, c for some $a, c \in V(G)$. Let $v_1, v_2, ..., v_m$ be vertices of $V(H^*)$ such that they are neighbors of x in G, and $b^*, v_1, v_2, ..., v_k, a^*$ appear in $C^* \cap H^*$ in this order. (See Fig. 4.52 and 4.53.)

Let P be the path of $F_{H^*} \cap C^*$ from v_k to b^*. By Corollary 3.2, there exists an F_{H^*}-Tutte trail T_{H^*} in H^* from v_k to b^* such that either $z \in V(T_{H^*})$ or the

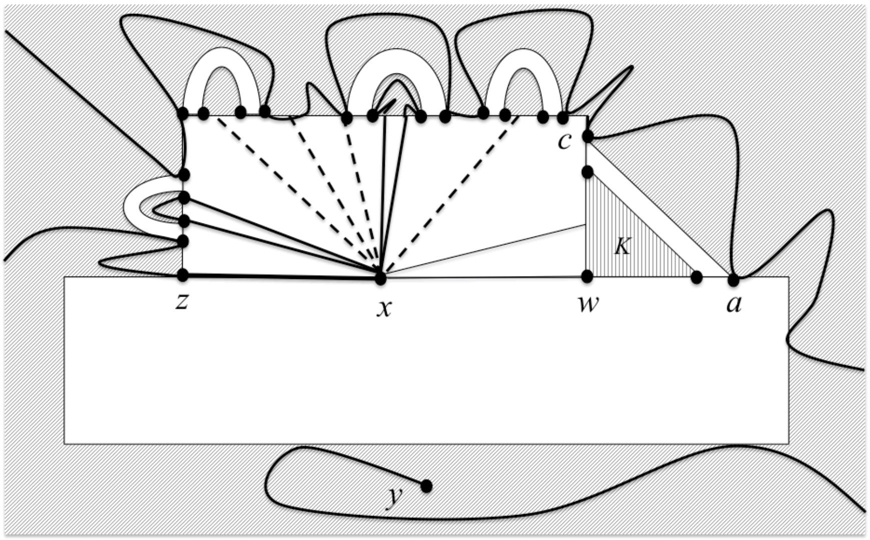


Figure 4.51: The structure of H and G in Case 5.1.2 when $w \notin V(T^*)$.

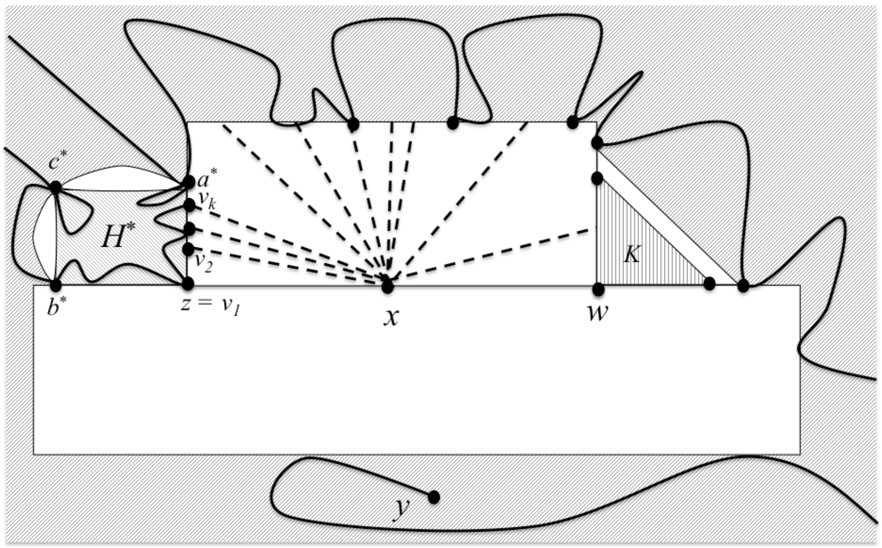


Figure 4.52: The structure of H^* and T'' in G in Case 5.2 when $b^* \in V(C)$.

component M of $G \backslash V(T_{H^*})$ containing z has exactly one edge connecting it to T. Since $z \in V(P)$, M cannot have only one edge connecting it to T_{H^*}. This implies that $z \in V(T_{H^*})$.

Since a^* is a vertex of F_{H^*}, $a^* \in V(T_{H^*})$ or the component B of $G \backslash V(T)$ con-

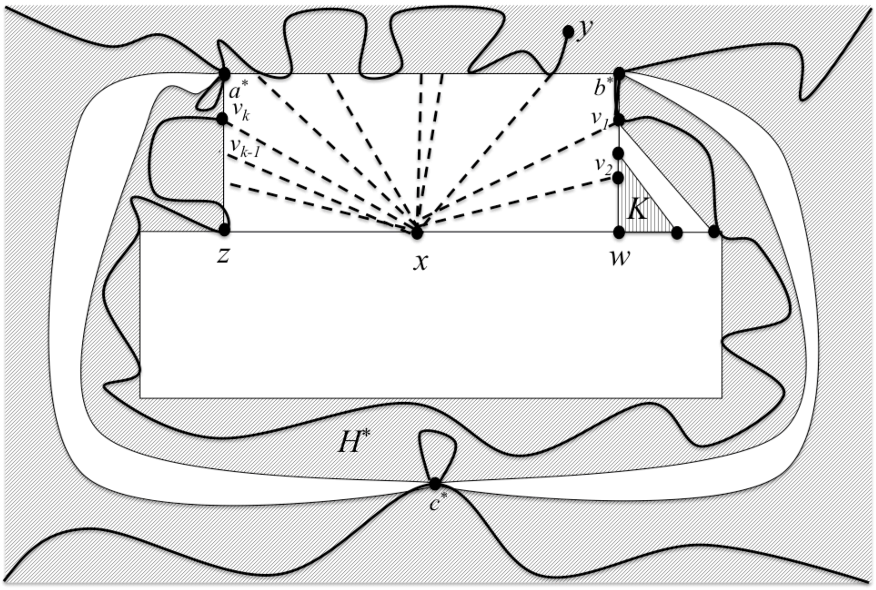


Figure 4.53: The structure of H^* and T'' in G in Case 5.2 when $b^* \in V(C^*)\backslash V(C)$.

taining a^* has at most two edges connecting it to T_{H^*}. Let A be the edge-block of B containing a^*, and $x_1, x_2 \in V(A)$ be such that $x_1 \neq a^*$, $x_2 \neq a^*$ and both vertices have a neighbor in $H^*\backslash A$. (Possibly $x_1 = x_2$.) By Theorem 3.1(a), A has an F_A-Tutte trail T_{a^*} from a^* to a^* containing x_1. (If $a^* \in V(T_{H^*})$, we let $T_{a^*} = \{a^*\}$.) If $x_2 \notin T_{a^*}$, the component of $A\backslash T_{a^*}$ containing x_2 has two edges connecting it to T_{a^*}, and then the component of $H^*\backslash V(T_{H^*} \cup T_{a^*})$ containing x_2 has three edges connecting it to $T_{H^*} \cup T_{a^*}$. Since c^* is a vertex of F_{H^*}, $c^* \in V(T_{H^*} \cup T_{a^*})$ or the component of $G\backslash V(T)$ containing c^* has at most two edges connecting it to $T_{H^*} \cup T_{a^*}$. Then we use exactly the same method again to get T_{c^*}. Hence $T' = T_{H^*} \cup T_{a^*} \cup T_{c^*}$ is the P-Tutte subgraph of H^*. (See Fig. 4.52 and 4.53.) Next, we will consider the trail $T'' = T^* \cup T'$ of G^*.

If $w \notin V(T'')$, we let K be the component of $G^*\backslash V(T'')$ containing w. Note that K is a plane graph since $V(K) \cap V(C^*) \neq \emptyset$. Let $J_1, J_2, ..., J_n$ be the distinct components of $G^*\backslash V(T'')$ such that $J_i \neq K$ and J_i contains a vertex of $V(C^*)\backslash V(C)$ for all $1 \leq i \leq n$. (Possibly, there is no such component.) Then we use the same method in Case 5.1.2 to define T_{J_i}.

If $w \in V(T'')$, we let $H = \{x\}$, and if $w \notin V(T'')$, we let H be the $(T'' \cup \{x\}$-

bridge of G containing w. Note that H is a C-flap with three vertices of attachment, one of which is x. Let a and c be the other two vertices of attachment of H such that $a \in V(C)$ and $c \in V(C^*)\backslash V(C)$. Note that $V(H) = V(K) \cup \{a, b, c\}$. (See Fig. 4.54 and 4.55.) Then H is the desired C-flap of G with attachments a, x, c and $T = T'' \cup \{v_k x\} \cup \bigcup_{i=1}^{n} T_{J_i}$ is the desired $(C\backslash(H\backslash\{a, x, c\}))$-Tutte trail of $G\backslash(H\backslash\{a, x, c\})$ from x to y.

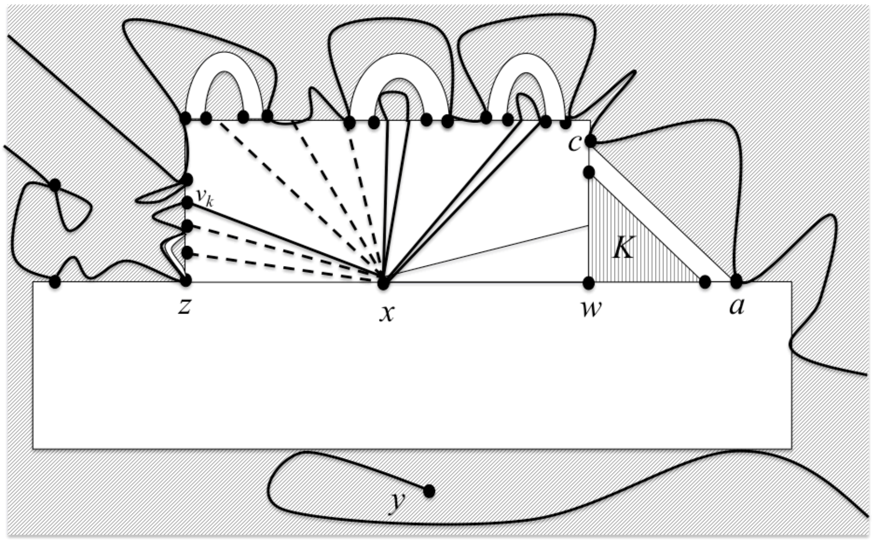


Figure 4.54: The structure of H and G in Case 5.2 when $w \notin V(T'')$ and $b^* \in V(C)$.

Case 5.3: H^* is non-trivial with $a^*, b^* \in V(C)$.
In this case, we let H be the subgraph of G induced by $V(H^*) \cup \{x\}$. Note that a^*, b^*, c^* are the vertices of attachment of H. Then H is the desired C-flap of G and T^* is also the desired $C\backslash(H\backslash\{a^*, b^*, c^*\}))$-Tutte trail of $G\backslash(H\backslash\{a^*, b^*, c^*\}$. (See Fig. 4.56.)

■

Let G be a 2-connected projective plane graph, R be a face of G, C be the facial walk of R, $x \in V(C)$ and $y \in V(G)\backslash\{x\}$. By Theorem 4.7, G has: a C-flap H with attachments $\{a, b, c\}$ containing x; and a trail T from b to y containing $\{a, c\}$. In Theorem 4.8, we show that G has a Tutte trail from x to y. We also extend the result from 2-connected graphs to 2-edge-connected graphs.

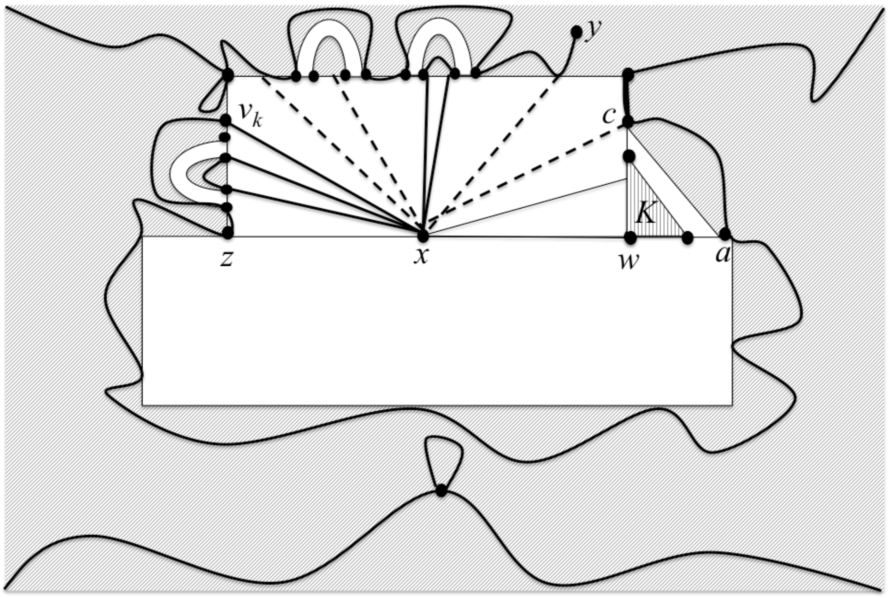


Figure 4.55: The structure of H and G in Case 5.2 when $w \notin V(T'')$ and $b^* \in V(C^*) \backslash V(C)$.

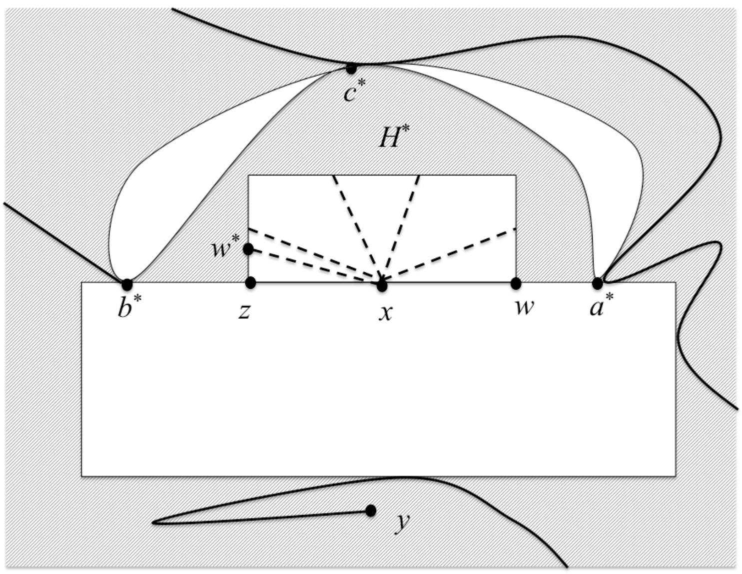


Figure 4.56: The structure of H and T^* in G in Case 5.3.

Theorem 4.8 *Let G be a 2-edge-connected graph embedded on the projective plane, R be a face of G, C be the facial walk of R, $x \in V(C)$ and $y \in V(G)\backslash\{x\}$. Then there exist:*

(i) a C-edge-flap H in G with attachments a, b, c such that $x, y \notin V(H)\backslash\{a, b, c\}$ and if H is trivial, we let $H = \{x\}$;

(ii) a $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail T in $G\backslash(H\backslash\{a, b, c\})$ from x to y such that $a, b, c \in V(T)$.

(iii) Moreover, every component of $G\backslash V(T)$ which contains an essential cycle is vertex-disjoint from C.

Proof. If G does not contain an essential cycle, then G is a plane graph and, by Theorem 3.1(a), G has a C-Tutte trail T from x to y. Then $H = \{x\}$ is the desired C-edge-flap and T is the desired C-Tutte trail of G. Hence we may assume that G contains an essential cycle. We prove the theorem by using induction on $|V(G)|$. If $|V(G)| = 2$, then we take $H = \{x\}$ and a trail $T = \{x, y, xy\}$ is the desired C-Tutte trail of G. So we assume $|V(G)| \geq 3$ and proceed to the induction step.

Case 1: G is 2-connected.

Let $w \in V(C)$ with $wx \in E(C)$. By Theorem 4.7, there exist: a C-flap H^* in G with attachments a^*, b^*, c^* such that $y \notin V(H^*)\backslash\{a^*, b^*, c^*\}$ and $x \in (V(H^*)\backslash\{a^*\}) \cup \{b^*\}$, and if H^* is non-trivial, then $w \in V(H^*)\backslash\{b^*\}$ and a^*, w, x, b^* appear in $C \cap H^*$ in this order; a $(C\backslash(H^*\backslash\{a^*, b^*, c^*\}))$-Tutte trail T^* in $G\backslash(H^*\backslash\{a^*, b^*, c^*\})$ from b^* to y such that $a^*, c^* \in V(T^*)$. Moreover, every component of $G\backslash V(T^*)$ which contains an essential cycle is vertex-disjoint from C.

If $H^* = \{x\}$, then $H = \{x\}$ is the desired C-edge-flap and T^* is the desired C-Tutte trail in G from x to y.

Hence we may assume that H^* is non-trivial. Let P be the path of $F_{H^*} \cap C$ from x to b such that $a, c \notin V(P)$. Then by Theorem 3.3, there exists a P-Tutte subgraph T of H^* consisting of three edge-disjoint trails T_{H^*}, T_{a^*} and T_{c^*} such that T_{H^*} is a trail from x to b^*, T_{a^*} and T_{c^*} are closed trails containing a^* and c^*, respectively, and $x \notin V(T_{a^*}) \cup V(T_{c^*})$, and there is at most one component D of $H^*\backslash V(T)$ which contains a vertex of F_{H^*} and has three edges connecting it to T.

If D does not exist, then $H = \{x\}$ is the desired C-edge-flap and $T^* \cup T$ is the desired C-Tutte trail in G from x to y.

Hence we may assume that D exists. Let H be the T-bridge of H^* containing D such that a, b, c are the vertices of attachment of H and $a, b \in V(C)$. (See Fig. 4.57.) Then H is desired C-edge-flap and $T^* \cup T$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$ from x to y.

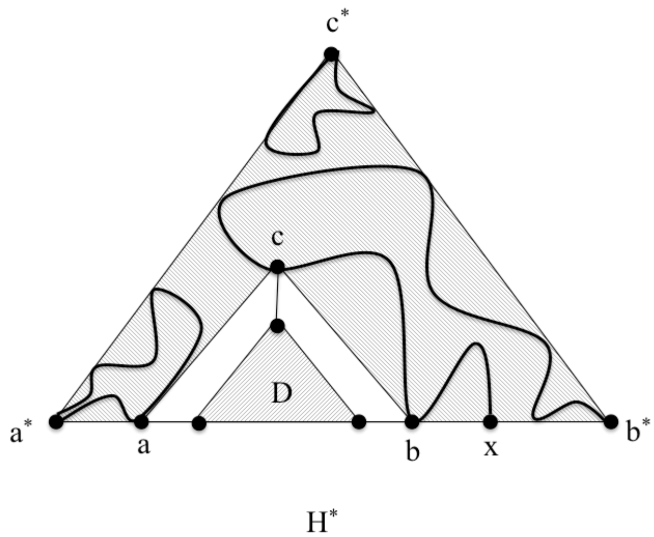


Figure 4.57: The structure of H^* in Case 1.

Case 2: G has a 1-separation (G_1, G_2) such that G_1 contains an essential cycle and G_2 is a 2-connected plane graph.
Let $z \in V(G_1) \cap V(G_2)$. Note that G_1 is 2-edge-connected. Let R^* be the face of G with its facial walk C^* such that $E(C^*) \cap E(G_i) \neq \emptyset$ for $i = 1, 2$. Let $C_i = C^* \cap G_i$ for $i = 1, 2$. Note that $z \in V(C_1) \cap V(C_2)$, C_1 is a facial walk in G_1, and C_2 is the outer walk in G_2.

Case 2.1: $R^* = R$.
Note that $C^* = C$. Let $x^* \in \{x, z\} \cap G_1$ and $y^* \in \{y, z\} \cap G_1$. By induction, there exist: a C_1-edge-flap H_1 in G_1 with attachments a, b, c such that $x^*, y^* \notin V(H_1) \backslash \{a, b, c\}$ and if H_1 is trivial, we let $H_1 = \{x^*\}$; a $(C_1 \backslash (H_1 \backslash \{a, b, c\}))$-Tutte trail T_1 in $G_1 \backslash (H_1 \backslash \{a, b, c\})$ from x^* to y^* such that $a, b, c \in V(T_1)$. Moreover, every component of $G_1 \backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C_1.

Subcase 2.1.1: $x, y \in V(G_1)$.
In this case, we let $x^* = x$ and $y^* = y$.

Suppose $z \in V(T_1)$. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2 from z to z. Then $H = H_1$ is the desired C-edge-flap of G, and $T_1 \cup T_2$ is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.58.)

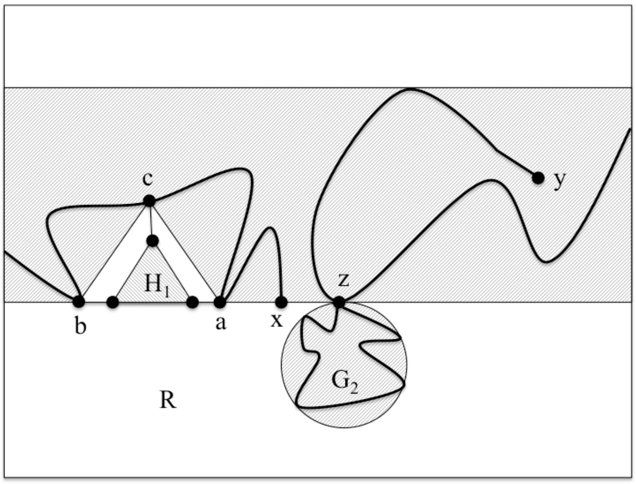


Figure 4.58: The structure of G in Case 2.1.1 when $z \in V(T_1)$.

Suppose $z \in V(H_1)\backslash\{a, b, c\}$. Note that both H_1 and G_2 are plane graphs. Then $H = H_1 \cup G_2$ is the desired C-edge-flap of G, and T_1 is the desired $(C\backslash(H\backslash\{a, b, c\}))$-Tutte trail in $G\backslash(H\backslash\{a, b, c\})$. (See Fig. 4.59.)

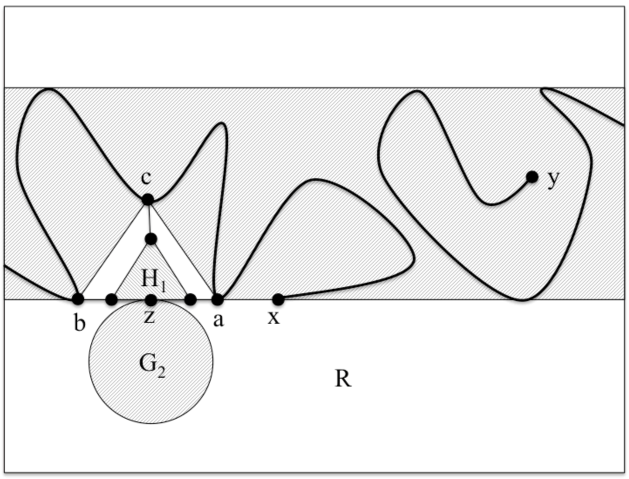


Figure 4.59: The structure of G in Case 2.1.1 when $z \in V(H_1)\backslash\{a, b, c\}$.

Suppose $z \notin V(H_1) \cup V(T_1)$. Let D be the component of $G_1 \backslash V(T_1)$ containing z. Since $z \in V(C_1)$, then D and $D \cup G_2$ have at most two edges connecting them to T_1. Then $H = H_1$ is the desired C-edge-flap of G, and T_1 is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig. 4.60.)

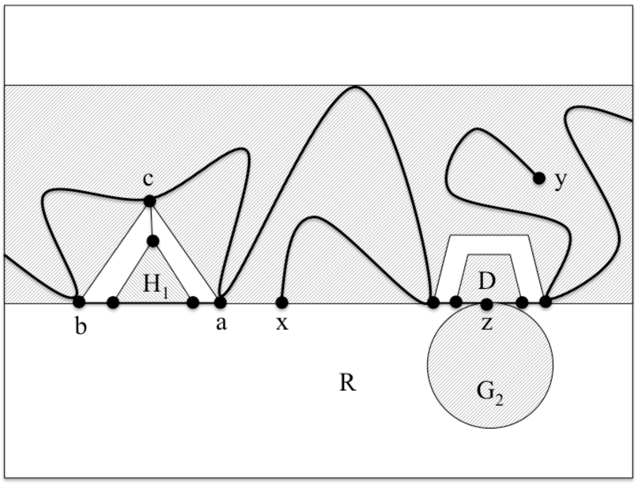


Figure 4.60: The structure of G in Subcase 2.1.1 when $z \notin V(H_1) \cup V(T_1)$.

Subcase 2.1.2: $x \in V(G_1)$ and $y \in V(G_2) \backslash \{z\}$.
In this case, we let $x^* = x$ and $y^* = z$. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2' from z to y. Then $H = H_1$ is the desired C-edge-flap of G, and $T_1 \cup T_2'$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig. 4.61.)

Subcase 2.1.3: $x \in V(G_2) \backslash \{z\}$ and $y \in V(G_1)$.
In this case we let $x^* = z$ and $y^* = y$. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2'' from x to z. Then $H = H_1$ is the desired C-edge-flap of G, and $T_1 \cup T_2''$ is the desired $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail in $G \backslash (H \backslash \{a, b, c\})$. (See Fig. 4.62.)

Subcase 2.1.4: $x \in V(G_2) \backslash \{z\}$ and $y \in V(G_2) \backslash \{z\}$.
By Thereom 4.4, there is a C_1-Tutte closed trail T of G_1 containing an edge of C_1 incident to z such that every component of $G_1 \backslash V(T)$ which contains an essential cycle is vertex-disjoint from C. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2''' from x to y containing z. Then $H = \{x\}$ is the desired C-edge-flap of G, and $T \cup T_2'''$ is the desired C-Tutte trail in G. (See Fig. 4.63.)

Case 2.2: $R^* \neq R$ and $C \subseteq V(G_1)$

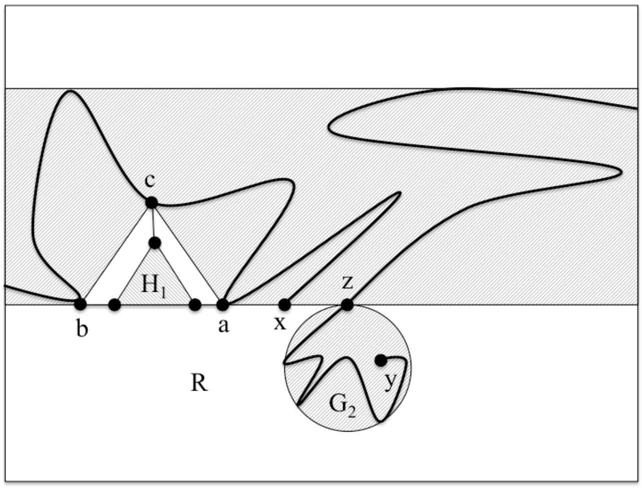


Figure 4.61: The structure of G in Subcase 2.1.2.

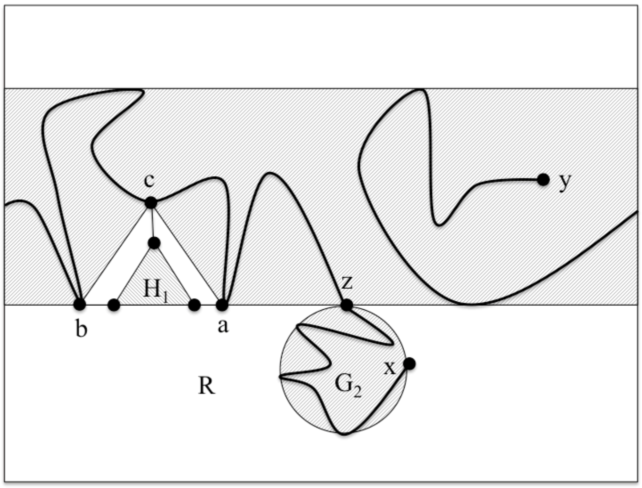


Figure 4.62: The structure of G in Subcase 2.1.3.

Let $y^* \in \{y, z\} \cap G_1$. By induction, there exist: a C-edge-flap H_1 in G_1 with attachments a, b, c such that $x, y^* \notin V(H_1)\backslash\{a, b, c\}$ and if H_1 is trivial, we let $H_1 = \{x^*\}$; a $(C\backslash(H_1\backslash\{a, b, c\}))$-Tutte trail T_1 in $G_1\backslash(H_1\backslash\{a, b, c\})$ from x to y^* such that $a, b, c \in V(T_1)$. Moreover, every component of $G_1\backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C.

Subcase 2.2.1: $y \in V(G_1)$.

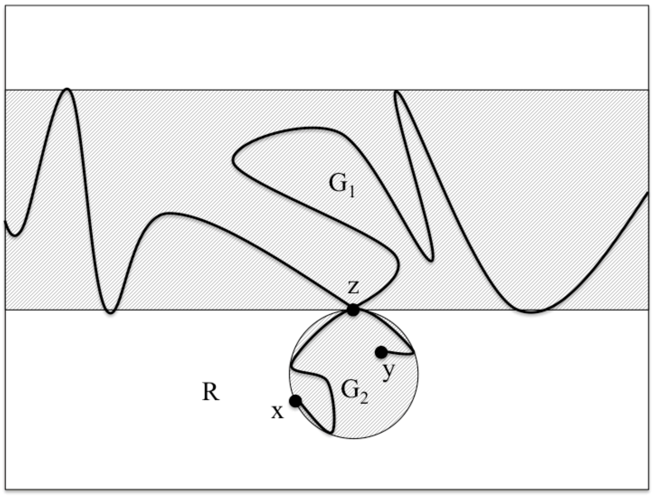


Figure 4.63: The structure of G in Subcase 2.1.4.

In this case, we let $y^* = y$.
Suppose $z \in V(T_1)$. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2 from z to z. Then $H = H_1$ is the desired C-edge-flap of G, and $T_1 \cup T_2$ is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.64.)

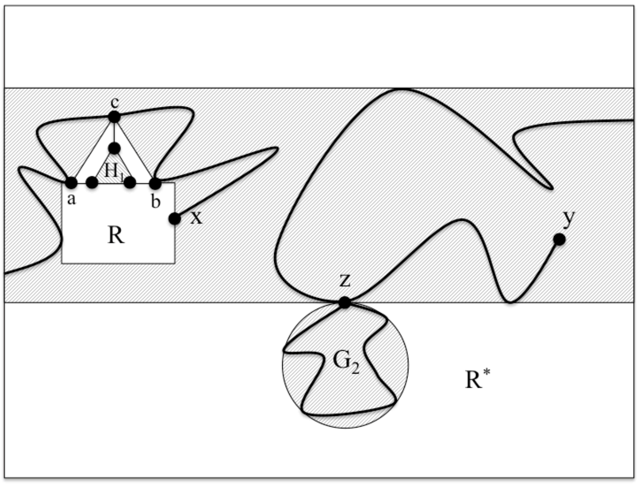


Figure 4.64: The structure of G in Subcase 2.2.1 when $z \in V(T_1)$.

Suppose $z \in V(H_1)\backslash\{a,b,c\}$. Note that both H_1 and G_2 are plane graphs. Then

$H = H_1 \cup G_2$ is the desired C-edge-flap of G, and T_1 is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.65.)

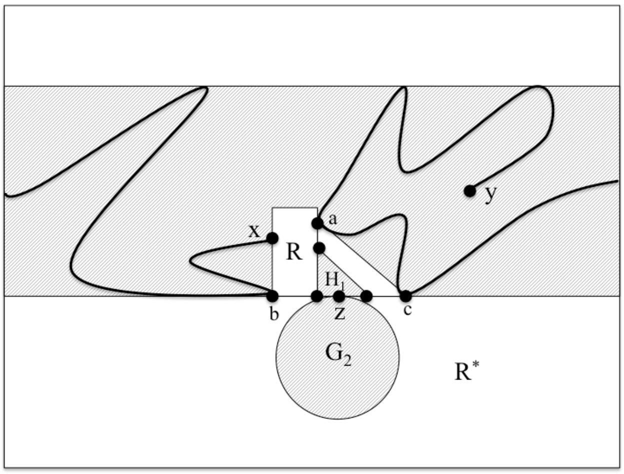


Figure 4.65: The structure of G in Subcase 2.2.1 when $z \in V(H_1)\backslash\{a,b,c\}$.

Suppose $z \notin V(H_1) \cup V(T_1)$. Let D be the component of $G_1\backslash V(T_1)$ containing z. Thus D and $D \cup G_2$ have at most three edges connecting them to T_1.(If D contains a vertex of C, then D and $D \cup G_2$ have at most two edges connecting them to T_1.) Then $H = H_1$ is the desired C-edge-flap of G, and T_1 is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.66.)

Subcase 2.2.2: $y \in V(G_2)\backslash\{z\}$.

In this case, we let $y^* = z$. By Theorem 3.1(a), G_2 has a C_2-Tutte trail T_2' from z to y. Then $H = H_1$ is the desired C-edge-flap of G, and $T_1 \cup T_2'$ is the desired $(C\backslash(H\backslash\{a,b,c\}))$-Tutte trail in $G\backslash(H\backslash\{a,b,c\})$. (See Fig. 4.67.)

Case 2.3: $R^* \neq R$ and $C \subseteq V(G_2)$.

We consider two subcases as follows.

Subcase 2.3.1: $y \in V(G_1)$.

By induction, there exist: a C_1-edge-flap H_1 in G_1 with attachments a_1, b_1, c_1 such that $z, y \notin V(H_1)\backslash\{a_1,b_1,c_1\}$ and if H_1 is trivial, we let $H_1 = \{z\}$; a $(C_1\backslash(H_1\backslash\{a_1,b_1,c_1\}))$-Tutte trail T_1 in $G_1\backslash(H_1\backslash\{a_1,b_1,c_1\})$ from z to y such that $a_1, b_1, c_1 \in V(T_1)$. Moreover, every component of $G_1\backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C_1.

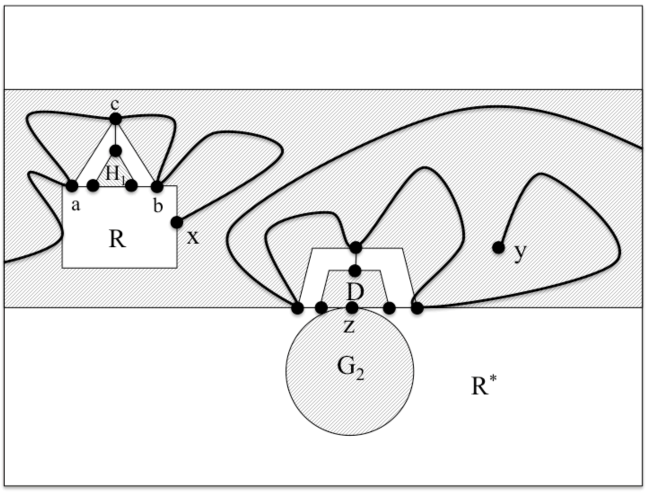


Figure 4.66: The structure of G in Subcase 2.2.1 when $z \notin V(H_1) \cup V(T_1)$.

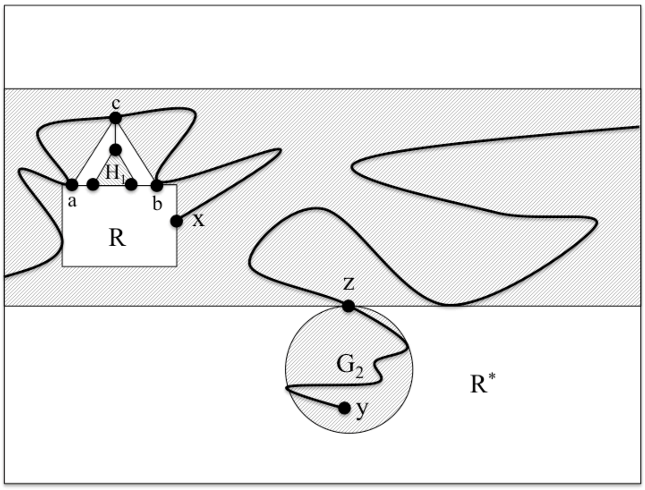


Figure 4.67: The structure of G in Subcase 2.2.2.

By Theorem 3.1(a), G_2 has a C-Tutte trail T_2 from x to z. Then $H = \{x\}$ is the desired C-edge-flap of G, and $T_1 \cup T_2$ is the desired C-Tutte trail in G. (See Fig. 4.68.)

Subcase 2.3.2: $y \in V(G_2)\backslash\{z\}$.

Since G_2 is 2-connected, by Lemma 4.5, G_2 has a C-Tutte trail T_2 from x to y such that either $z \in V(T_2)$, or $z \notin V(T_2)$ and the component D of $G_2\backslash V(T_2)$ containing

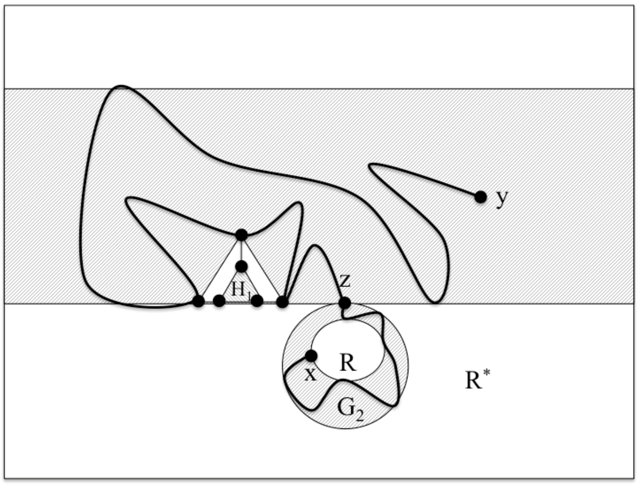


Figure 4.68: The structure of G in Subcase 2.3.1.

z is vertex-disjoint from C.

Suppose $z \in V(T_2)$. Then by Theorem 4.4, there is a C_1-Tutte closed trail T_1 of G_1 containing an edge of C_1 incident to z such that every component of $G_1 \backslash V(T_1)$ which contains an essential cycle is vertex-disjoint from C. Then $H = \{x\}$ is the desired C-edge-flap of G, and $T_1 \cup T_2$ is the desired C-Tutte trail in G. (See Fig. 4.69.)

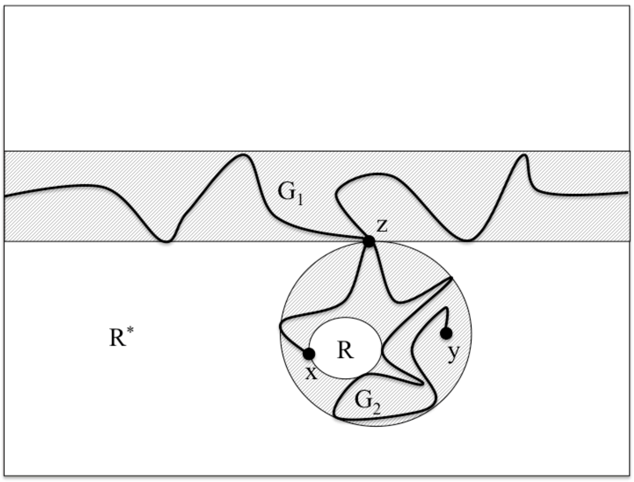


Figure 4.69: The structure of G in Subcase 2.3.2 when $z \in V(T_2)$.

Suppose $z \notin V(T_2)$. Then $H = \{x\}$ is the desired C-edge-flap of G, and T_2 is the desired C-Tutte trail in G. (See Fig. 4.70.) Note that Since D is vertex-disjoint from C, then $D \cup G_1$ is also vertex-disjoint from C.

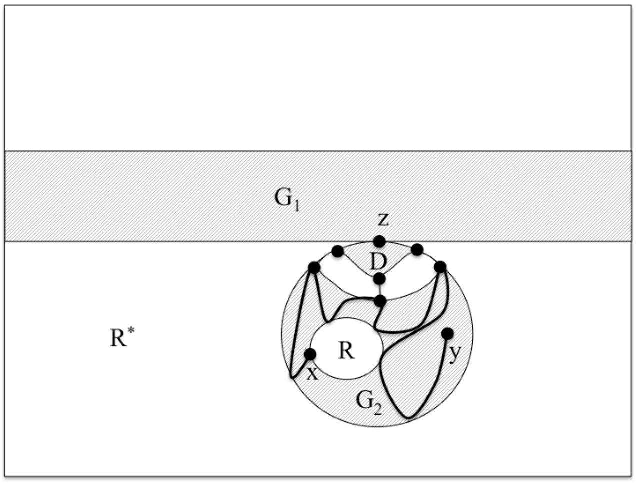


Figure 4.70: The structure of G in Case 2.3.2 when $z \notin V(T_2)$.

Case 3: G has a 1-separation (G_1, G_2) such that both G_1 and G_2 contain an essential cycle, and there is no 1-separation of G as in Case 2.
Let $z \in V(G_1) \cap V(G_2)$. In this case, we will show that $H = \{x\}$ is the desired C-edge-flap of G. Note that the representativity of G is one and all blocks of G contain an essential cycle and z. Let $B_1, B_2, ..., B_n$ be the blocks of G. Assume without the loss of generality that $x \in V(B_1)$. Let R^* be a face of G such that the R^*-width of G is one, and let C^* be the facial walk of R^*. (If $R = R^*$, we assume that $E(C^*) \cap E(B_1) \neq \emptyset$ and $E(C^*) \cap E(B_2) \neq \emptyset$.) (See Fig. 4.71.)

Let R_i be the face of B_i such that the R_i-width of B_i is one, and let C_i be the facial walk of R_i for $1 \leq i \leq n$. Then for all $1 \leq i \leq n$, B_i can be redrawn as a plane graph B_i^* such that $C_i = C_i^* \cup D_i^*$ where C_i^* is the outer cycle of B_i^*, D_i^* is another facial cycle of B_i^*, and $z \in C_i^* \cap D_i^*$. (See Fig. 4.72.) Note that B_i^* is 2-connected. Then by Theorem 3.1(a), B_i^* has a C_i^*-Tutte trail T_i from z to z for all $1 \leq i \leq n$.

Case 3.1: $R = R^*$.
Assume without the loss of generality that $x \in C_1^*$ and $E(C_i^*) = E(B_i) \cap E(C)$ for $i = 1, 2$. Suppose $y \in V(B_1)$. Then by Theorem 3.1, B_1^* has a C_1^*-Tutte T^*

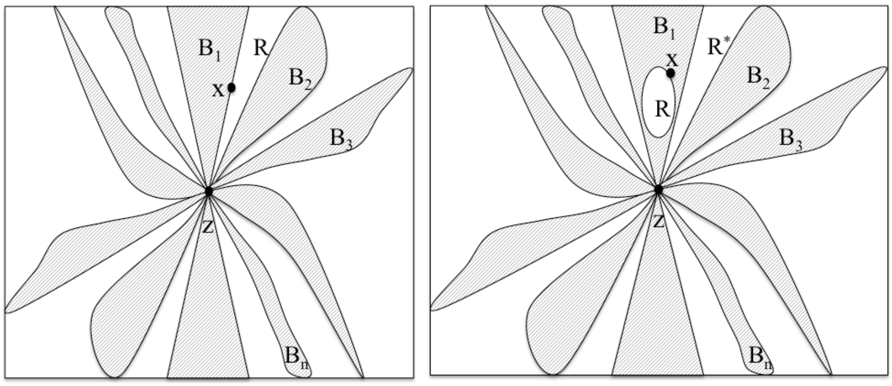


Figure 4.71: The structure of G in Case 3 when the R-width of G is one(left) and the R-width of G is more than one (right).

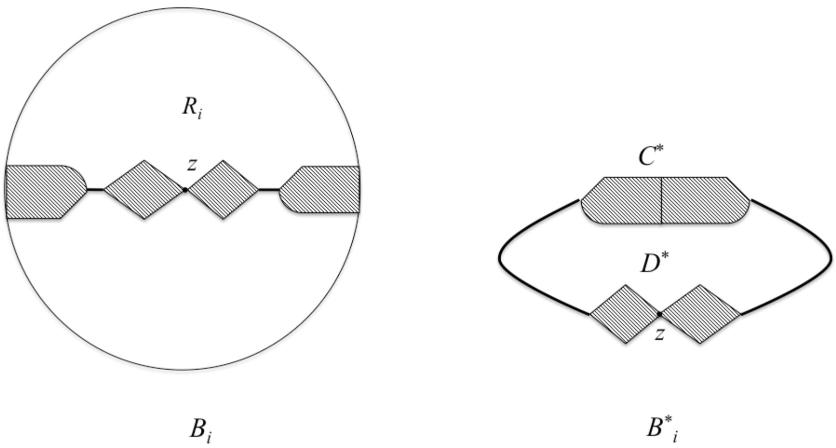


Figure 4.72: The structure of B_i and B_i^* in Case 3.1

trail from x to y containing z. Then $H = \{x\}$ is the desired C-edge-flap of G, and $T^* \cup \bigcup_{i=2}^{n} T_i$ is the desired C-Tutte trail in G. (See Fig. 4.73.)

Suppose $y \in V(B_j)$ for some $2 \leq j \leq n$. Then by Theorem 3.1(a), B_1^* has a C_1^*-Tutte T_1' trail from x to z, and B_j^* has a C_j^*-Tutte trail T_j' from z to y. Then $H = \{x\}$ is the desired C-edge-flap of G, and $T_1' \cup T_j' \cup \bigcup_{i=2,i\neq j}^{n} T_i$ is the desired C-Tutte trail in G. (See Fig. 4.74.)

Case 3.2: $R \neq R^*$.

Then the R-width of G is more than one. Since $x \in V(B_1)$, $C \subseteq B_1$.

Subcase 3.2.1: $y \in V(B_1)$.

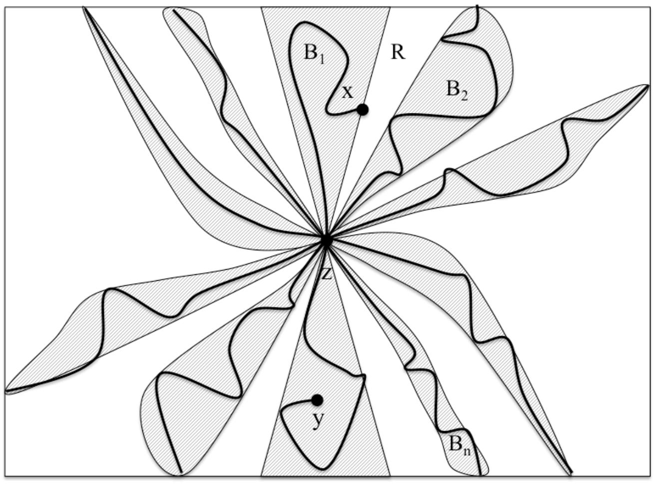


Figure 4.73: The structure of B_i and B_i^* in Case 3 when $y \in V(B_1)$.

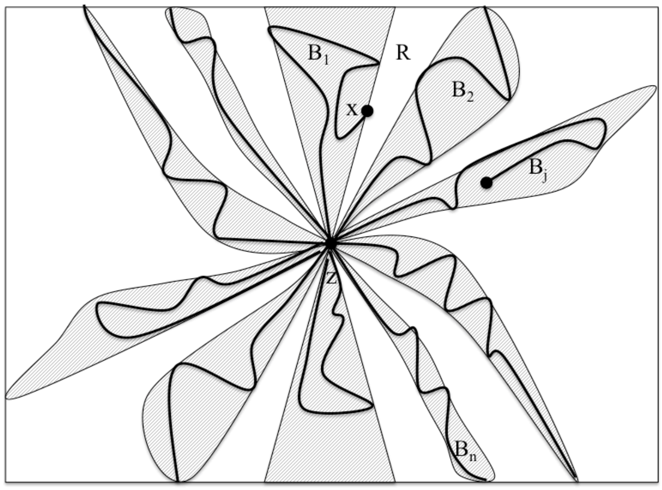


Figure 4.74: The structure of G in Case 3.1 when $y \in V(B_j)$ for some $2 \leq j \leq n$.

Then by Lemma 4.5, B_1^* has a C-Tutte trail T^* from x to y such that either $z \in V(T^*)$, or $z \notin V(T^*)$ and the component U of $B_1^* \backslash V(T^*)$ containing z is vertex-disjoint from C.

If $z \in V(T^*)$, then $H = \{x\}$ is the desired C-edge-flap of G, and $T^* \cup \bigcup_{i=2}^{n} T_i$ is the desired C-Tutte trail in G. (See Fig. 4.75.)

If $z \notin V(T^*)$, then $H = \{x\}$ is the desired C-edge-flap of G, and T^* is the desired C-Tutte trail in G. (See Fig. 4.76.) Note that the component $U \cup \bigcup_{i=2}^{n} B_i$

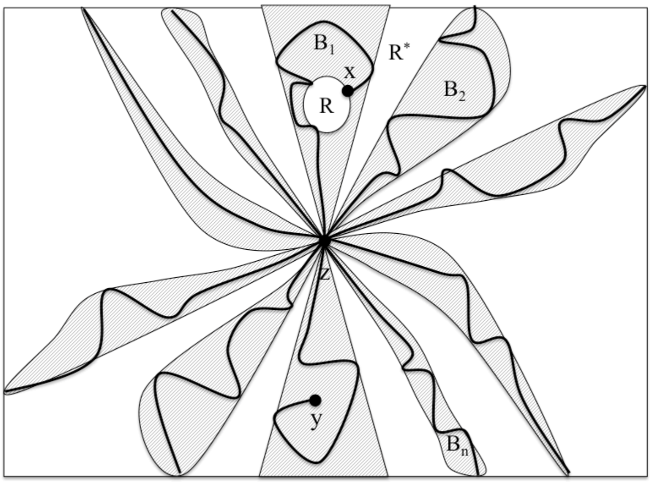


Figure 4.75: The structure of G in Subcase 3.2.1 when $z \in V(T^*)$.

of $G \backslash V(T^*)$ is vertex-disjoint from C since U is vertex-disjoint from C.

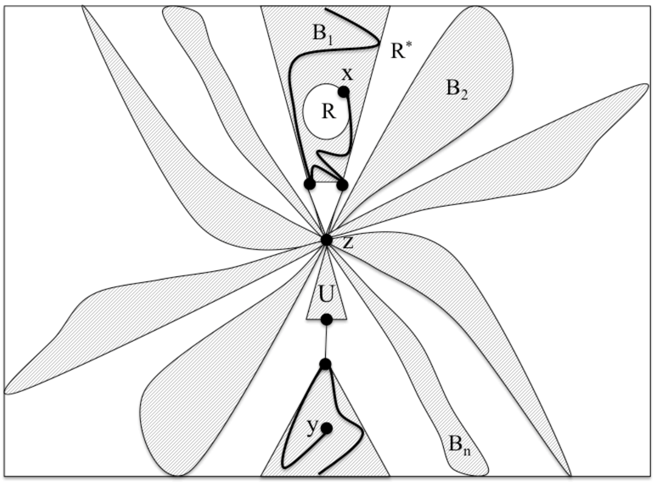


Figure 4.76: The structure of G in Subcase 3.2.1 when $z \notin V(T^*)$.

Subcase 3.2.2: $y \in V(B_j)$ for some $2 \leq j \leq n$.
Then by Theorem 3.1, B_1^* has a C-Tutte trail T_1'' from x to z, and B_j^* has a a C_j^*-Tutte trail T_j' from z to y. Then $H = \{x\}$ is the desired C-edge-flap of G, and $T_1'' \cup T_j' \cup \bigcup_{i=2,i\neq j}^{n} T_i$ is the desired C-Tutte trail in G. (See Fig. 4.77.)

■

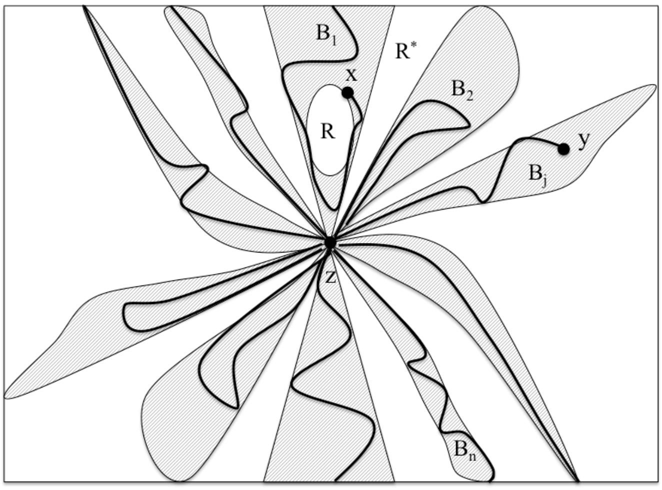


Figure 4.77: The structure of G in Subcase 3.2.2.

Chapter 5

Tutte Closed Trail of Toroidal Graphs

The main result of this chapter is that every 2-edge-connected toroidal graph with representativity at most two has a Tutte closed trail.

We prove the following lemma.

Lemma 5.1 *Let G be a 2-edge-connected toroidal plane graph with representativity at least two. If G has a 1-separation (G_1, G_2) such that the representativity of G_1 is at least two, then G_2 does not contain an essential cycle.*

Proof. Suppose that G_2 has an essential cycle C. Since the representativity of G_1 is at least two, every essential closed curve in the torus must intersect G_1 at least twice. Since we can consider C as an essential closed curve, $|V(C) \cap V(G_1)| \geq 2$. This is a contradiction because $|V(G_1) \cap V(G_2)| = 1$. Hence G_2 does not contain an essential cycle. ■

Kawarabashi and Ozeki [17] showed the following lemma.

Lemma 5.2 *Let G be a 2-connected graph on the torus with representativity exactly 2, let R be a face of G, let C be the facial walk of R, and let $x \in V(C)$. Suppose that the (x, R)-width of G is exactly 2. Then G can be decomposed into three plane graphs G_0, B and D such that G_0 is 2-connected, $B = G_{S_1}$ where S_1 is a chain of blocks $b_0, B_1, b_1, B_2, ..., b_{n-1}, B_n, b_n$, and $D = G_{S_2}$ where S_2 is a chain of*

blocks $d_0, D_1, d_1, D_2, ..., d_{m-1}, D_m, d_m$ satisfying properties (G1) to (G4). (Possibly, $|V(B)| = 1$ and/or $|V(D)| = 1$, in this case we take $b_0 = b_n$ and/or $d_0 = d_m$.)
(G1) There exist two vertex-disjoint facial cycles C_1 and C_2 in G_0 and four distinct vertices u_1, u_2, v_1 and v_2 such that $u_1, v_1 \in V(C_1)\backslash V(C_2)$ and $u_2, v_2 \in V(C_2)\backslash V(C_1)$.
(G2) G is obtained from G_0, B and D by identifying u_1 and b_0, u_2 and b_n, v_1 and d_0, and v_2 and d_m, respectively.
(G3) $E(C) = E(C_1[u_1, v_1] \cup C_2[v_2, u_2]) \cup \bigcup_{i=1}^{n} E(F_{B_i}[b_i, b_{i-1}]) \cup \bigcup_{i=1}^{m} E(F_{D_i}[d_i, d_{i-1}])$.
(G4) $x = d_j$ for some $0 \leq j \leq m$. (See Fig. 5.1.)

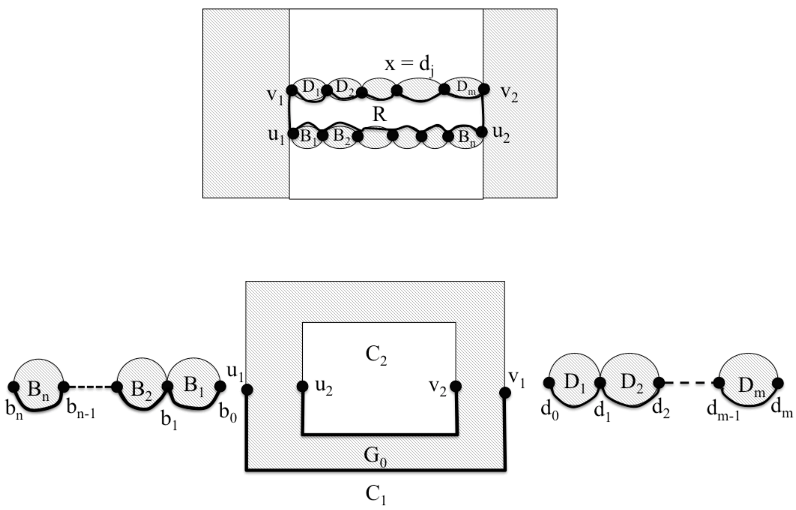


Figure 5.1: The structure of G in Lemma 5.2.

We use Theorem 3.7 and Lemma 5.2 to prove the following theorem.

Theorem 5.3 *Let G be a 2-edge-connected graph embedded on torus with representativity at most two. Then G has a Tutte closed trail.*

Proof. We prove this using induction on $|V(G)|$. If $V(G) = \{x, y\}$, then $T = \{x, y, xy\}$ is the desired Tutte closed trail of G. So we assume $|V(G)| \geq 3$ and proceed to the induction step. We consider cases depending on the representativity of G.

Case 1: The representativity of G is zero.
Then G can be redrawn as a 2-edge-connected plane graph G^*. By Theorem 3.1(a),

G^* has a Tutte trail T^* from x to x for any $x \in V(G)$. Then T^* is the desired Tutte closed trail of G.

Case 2: The representativity of G is one.

Let ϕ be an essential closed curve intersecting G only at a vertex x. Let $G_1, G_2, ..., G_n$ be the x-bridges of G. Note that these bridges can be labeled such that the representativity of G_1 is at most one, and the representativity of G_i is zero for $2 \leq i \leq n$. (See Fig. 5.2.) For $1 \leq i \leq n$, let G_i' be the graph obtained by cutting G_i along ϕ,

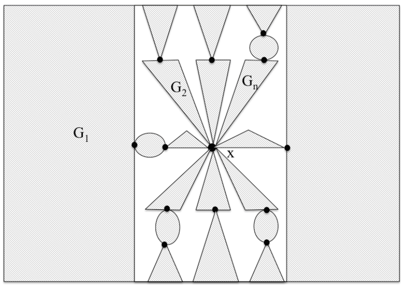


Figure 5.2: The structure of G in Case 2.

in which the vertices corresponding to x are x_{1i} and x_{2i}. (See Fig. 5.3.) Note that the representativity of G_i' is zero. Then G_i' can be redrawn as a connected plane

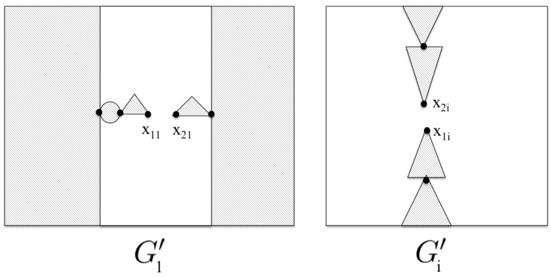


Figure 5.3: The structure of G_1' and G_i' in Case 2.

graph G_i^*. By Theorem 3.2, G_i^* has a Tutte trail T_i from x_{1i} to x_{2i}. By identifying x_{1i} and x_{2i}, T_i becomes a Tutte closed trail of G_i. Then $T = \bigcup_{i=1}^{n} T_i$ is the desired Tutte closed trail of G. (See Fig. 5.4.)

Case 3: The representativity of G is two.

Note that if $|V(G)| \leq 3$, then the representativity of G is at most one. Then, in this case, $|V(G)| \geq 4$.

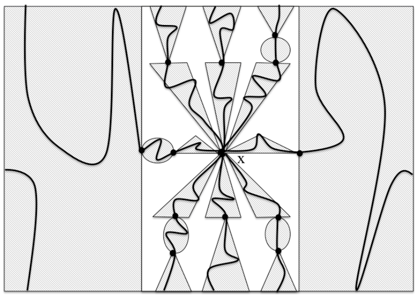


Figure 5.4: The structure of T in Case 2.

Case 3.1: G is 2-connected.

By Lemma 5.2, G can be decomposed into three plane graphs G_0, B and D as in Fig. 5.1. By Theorem 3.7, G_0 has an $F_{G_0}[u_1, v_1]$-Tutte subgraph T consisting of two edge-disjoint trails T_1 from u_1 to v_1, and T_2 from u_2 to v_2. By Corollary 3.2, B has an F_B-Tutte trail T_B from u_1 to v_1, and D has an F_D-Tutte trail T_D from u_2 to v_2. Then $T^* = T \cup T_B \cup T_D$ is the desired Tutte closed trail of G. (See Fig. 5.5.)

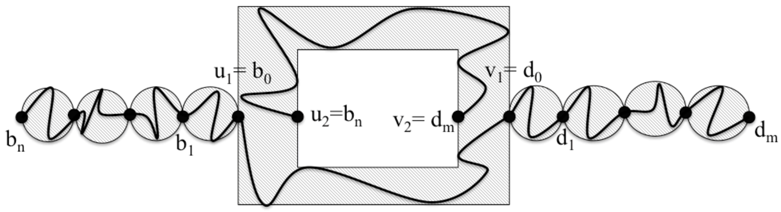


Figure 5.5: The structure of T^* in Case 3.1.

Case 3.2: G is not 2-connected.
By Lemma 5.1, G has 1-separation (G_1, G_2) of G such that $V(G_1) \cap V(G_2) = \{x\}$, the representativity of G_1 is two, G_2 does not contain an essential cycle, and note that both G_1 and G_2 are 2-edge-connected. By induction, G_1 has a Tutte closed trail T_1.

Suppose $x \notin V(T_1)$. Then the component D of $G_1 \backslash V(T_1)$ containing x has at most three edges connecting it to T_1. Then T_1 is the desired Tutte closed trail of G. (See Fig. 5.6.)

Suppose $x \in V(T_1)$. By Theorem 3.1(a), G_2 has an F_{G_2}-Tutte trail T_2 from x to x. Then $T_1 \cup T_2$ is the desired Tutte closed trail of G. (See Fig. 5.7.) ■

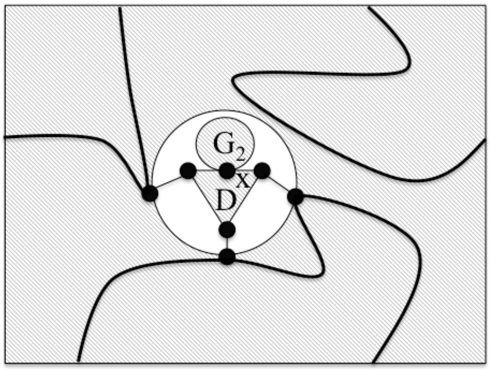


Figure 5.6: The structure of T_1 in Case 3.2 when $x \notin V(T_1)$.

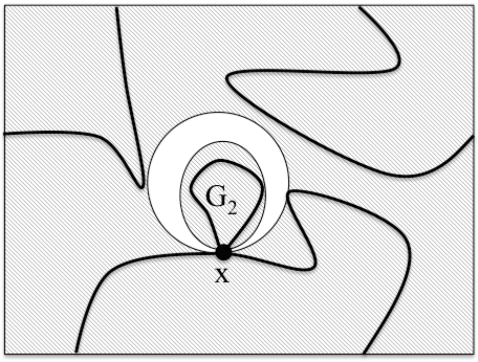


Figure 5.7: The structure of $T_1 \cup T_2$ in Case 3.2 when $x \in V(T_1)$.

Chapter 6

Conclusions and Future Work

6.1 Conclusions

The main aim of this thesis is to show that every 2-edge-connected graph embedded on a surface Σ has a Tutte closed trail. In Chapter 3 and 4, we show that a 2-edge-connected graph G embedded on plane or projective plane has a Tutte trail from x to y for any $x, y \in V(G)$. In Chapter 5, we show that a 2-connected toroidal graph G has a Tutte closed trail when the representativity of G is at most two. For the case of a toroidal graph with representativity more than two, we will need more results on plane graph to complete the proof.

We think that it may be possible to show "a 2-edge-connected graph embedded on a surface Σ has a Tutte closed trail" by using an inductive proof on the genus of the surface.

6.2 Dominating trails in essentially-4-edge-connected graphs

We also obtain results of dominating trail in essentially-4-edge-connected graphs by using the following theorem.

Theorem 6.1 *Let G be an essentially-4-edge-connected graph, and $x, y \in V(G)$. If G has a Tutte trail T from x to y, then T is a dominating trail of G from x to y.*

Proof. Since T is a Tutte trail from x to y, all components D of $G \backslash V(T)$ have at most three edges connecting it to T. Since G is essentially-4-edge-connected, D has one vertex. Then T is a dominating trail of G from x to y. ■

Using Theorem 6.1 and Theorem 3.1, we have the following theorem.

Theorem 6.2 *Let G be an essentially-4-edge-connected plane graph and F_G be the outer walk of G.*
(a) If $u, v \in V(G)$ and $e \in E(F_G)$ such that both end vertices of e have degree at least two, then there is a dominating trail in G from u to v containing e.
(b) If $|E(F_G)| \geq 3$ and $e_1, e_2, e_3 \in E(F_G)$ such that all end vertices of $\{e_1, e_2, e_3\}$ have degree at least two, then there is a dominating closed trail in G containing e_1, e_2 and e_3.
(c) If $E(F_G) = \{e_1, e_2\}$ such that all end vertices of $\{e_1, e_2\}$ have degree at least two, then there is a dominating closed trail closed trail in G containing e_1 and e_2.

Using Theorem 6.1, Theorem 4.4, and Theorem 4.7, we have the following theorem.

Theorem 6.3 *Let G be an essentially-4-edge-connected projective plane graph and C be a facial walk of G.*
(a) If $e \in E(C)$ such that both end vertices of e have degree at least two, then there is a dominating closed trail in G containing e.
(b) If $x \in V(C)$ and $y \in V(G) \backslash \{x\}$, then there is a dominating trail T in G from x to y.

Using Theorem 6.1 and Theorem 5.3, we have the following theorem.

Theorem 6.4 *Let G be an essentially-4-edge-connected toroidal graph with representativity at most two. Then G has a dominating closed trail.*

Harary and Nash-Williams [9] showed that the line graph $L(G)$ of G has a Hamilton cycle if and only if G has a dominating closed trail. Since $L(G)$ is k-connected if and only if G is essentially k-connected, Theorems 6.2, 6.3, and 6.4 imply the following theorem.

Theorem 6.5 *Let G be a plane graph, a projective plane graph, or a toroidal graph with representativity at most two. If the line graph $L(G)$ of G is 4-connected, then $L(G)$ has a Hamilton cycle.*

6.3 Future work and Conjectures

We will give some conjectures on Tutte subgraphs of plane graphs, projective planes, and toroidal graphs in the following subsections.

6.3.1 Plane graphs

We give the conjecture which is similar to Theorem 3.1(a) as follows.

Conjecture 6.6 *Let G be a 2-edge-connected plane graph, C_1 and C_2 be facial walks of G such that $x, y \in V(C_1)$ and $e \in V(C_2)$. If C^* is a subwalk of C_1 from x to y, then there exists a $(C^* \cup C_2)$-Tutte trail in G from x to y containing e.*

If Conjecture 6.6 is true, then it would imply the following conjecture which is similar to Theorem 3.7.

Conjecture 6.7 *Let G be a 2-connected plane graph, C_1 and C_2 be distinct facial walks of G such that $u_1, v_1 \in V(C_1)$ and $u_2, v_2 \in V(C_2)$. Then there exists a $(C_1[u_1, v_1] \cup C_2[u_2, v_2])$-Tutte subgraph in G consisting of two trails T_1 from u_1 to v_1, and T_2 from u_2 to v_2.*

We also give another possible conjecture which is similar to Theorem 3.7 as follows.

Conjecture 6.8 *Let G be a 2-connected plane graph, C_1 and C_2 be vertex-disjoint facial walks of G. Suppose $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$ are subsets of $V(G)$ such that x_1, x_2, x_3 appear in order on C_1, and y_1, y_2, y_3 appear in order on C_2. Then there exists a $C_1[x_1, x_2]$-Tutte subgraph T in G such that T satisfies exactly one of the following:*
(i) T is a closed trail containing X and Y;
(ii) $T = T_1 \cup T_2$ such that T_1 is a trail from x_1 to u_j containing u_k where $\{j, k\} = \{2, 3\}$ and $\{u_m, v_m\} = \{x_m, y_m\}$ for $m = j, k$, and T_2 is a trail from v_k to y_1

containing v_j;
(iii) $T = T_1 \cup T_2 \cup T_3$ such that T_1 is a trail from x_1 to x_j for some $j = 2, 3$, T_2 is a trail from y_j to y_k where $k \notin \{1, j\}$, and T_3 is a trail from x_k to y_1;
(iv) $T = T_1 \cup T_2 \cup T_3$ such that T_1 is a trail from x_1 to y_j for some $j = 2, 3$, T_2 is a trail from x_j to y_k where $k \notin \{1, j\}$, and T_3 is a trail from x_k to y_1.

It may be possible to extend this conjecture to the case when $|X| = |Y| = n$ for all $n \geq 4$.

6.3.2 Projective plane graphs

We give a conjecture which extends Theorem 4.7 by adding an edge e.

Conjecture 6.9 *Let G be a 2-edge-connected graph embedded on the projective plane, R be a face of G and C be the facial walk of R, $x \in V(C)$ $e \in E(C)$, and $y \in V(G) \backslash \{x\}$. Then there exist:*
(i) a C-edge-flap H in G with attachments a, b, c such that $x, y \notin V(H) \backslash \{a, b, c\}$ and if H is trivial, we let $H = \{x\}$;
(ii) a $(C \backslash (H \backslash \{a, b, c\}))$-Tutte trail T in $G \backslash (H \backslash \{a, b, c\})$ from x to y containing e such that $a, b, c \in V(T)$.
(iii) Moreover, every component of $G \backslash V(T)$ which contains an essential cycle is vertex-disjoint from C.

This conjecture would be useful to show that a 2-edge-connected graph embedded on the Klein bottle has a Tutte closed trail.

6.3.3 Toroidal graphs

To show that every 2-edge-connected toroidal graph has a Tutte closed trail, it remains to show the case when G has representativity at least three. If Conjecture 6.7 is true, it will extend Theorem 5.3 as follows.

Conjecture 6.10 *Let G be a 2-edge-connected toroidal graph with representativity at most two, R be a face of G, and C be the facial walk of R. If the R-width of G is at least two, then G has a C-Tutte closed trail.*

We think it may be possible to extend this conjecture to the case when the representativity of G is at least three.

Let G be a 2-connected toroidal graph. Kawarabashi and Ozeki showed in Theorem 5.2 that if the representativity of G is two, then G can be decomposed into three plane graphs satisfying some properties. So we also give another possible conjecture when the representativity G is n, for $n \geq 2$, as follows.

Conjecture 6.11 *Let G be a 2-connected graph on the torus with representativity n, for $n \geq 2$, R be a face of G, C be the facial walk of R, and $x \in V(C)$. Suppose that the (x, R)-width of G is n. Then G can be decomposed into $n+1$ plane graphs $G_0, B_1, B_2, ..., B_{n-1}$ and B_n such that G_0 is 2-connected, and for each $1 \leq i \leq n$, $B_i = G_{S_i}$ where S_i is a chain of blocks $a_0^i, A_1^i, a_1^i, A_2^i, ..., a_{n_i-1}^i, A_{n_i}^i, a_{n_i}^i$ satisfying properties (G1) to (G4). (Possibly, $|V(B_i)| = 1$ for some $1 \leq i \leq n$, and in this case we take $a_0^i = a_{n_i}^i$.)*
(G1) There exist two vertex-disjoint facial cycles C_1 and C_2 in G_0, and two distinct vertex set $U = \{u_1, u_2, ..., u_n\}$ and $V = \{v_1, v_2, ..., v_n\}$ such that $U \subseteq V(C_1)$, $V \subseteq V(C_2)$, $u_1, u_2, ..., u_n$ appear in order in C_1, and $v_1, v_2, ..., v_n$ appear in order on C_2.
(G2) G is obtained from $G_0, B_1, B_2, ..., B_{n-1}$ and B_n by identifying u_i and a_0^i, v_i and $a_{n_i}^i$ for each $1 \leq i \leq n$
(G3) $E(C) = E(C_1[u_1, u_2] \cup C_2[v_2, v_1]) \cup \bigcup_{i=1}^{n_1} E(F_{A_i^1}[a_i^1, a_{i-1}^1]) \cup \bigcup_{i=1}^{n_2} E(F_{A_i^2}[a_i^2, a_{i-1}^2])$.
(G4) $x = a_j^1$ for some $0 \leq j \leq n_1$. (See Fig. 6.1.)

If both Conjecture 6.8 and Conjecture 6.11 are true, then it would imply the following conjecture which is similar to Theorem 5.3 with the representativity changed from two to three.

Conjecture 6.12 *Let G be a 2-edge-connected toroidal graph with representativity three. Then G has a Tutte closed trail.*

6.3.4 Barnette 's Conjecture

Barnette [2] conjectured that every 3-connected cubic bipartite plane graph is Hamiltonian. We will consider the question: does every 2-edge-connected bipartite plane graph G have a G-Tutte closed trail ? An affirmative answer would imply that "every 3-edge-connected bipartite plane graph has a spanning closed trail" and hence

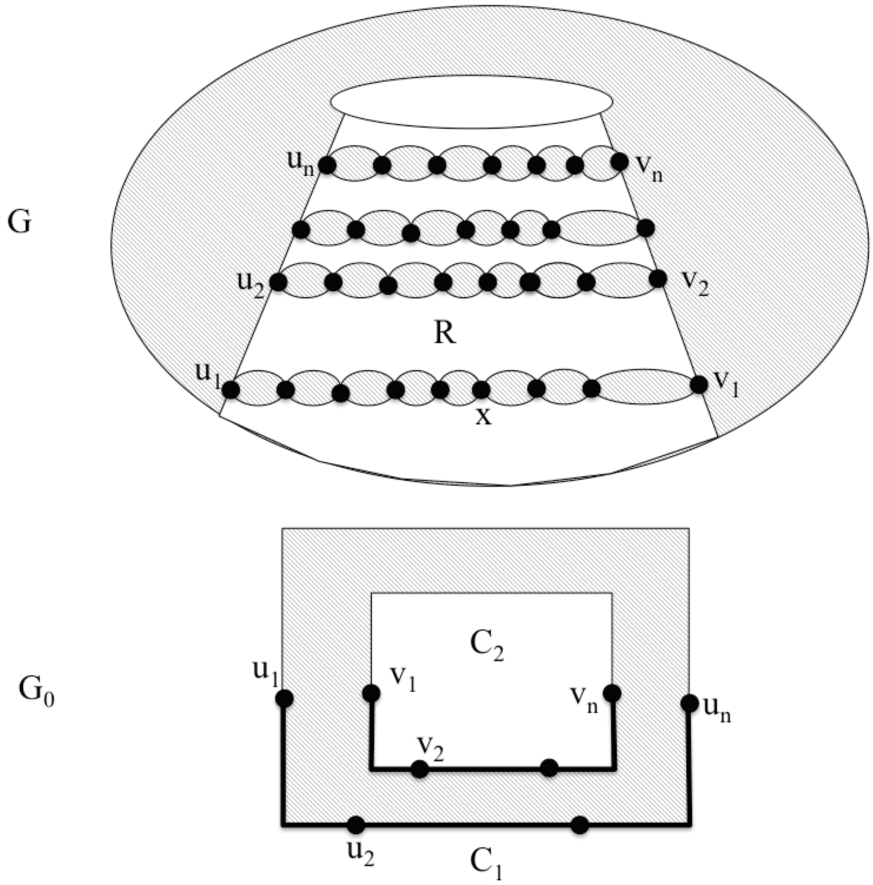


Figure 6.1: The structure of G and G_0 in Conjecture 6.11.

would imply Barnett's Conjecture. But the following counterexample shows that the answer is negative. Let G be a non-Hamiltonian 3-connected cubic plane graph (e.g. Tutte graph). Let G^* be a graph obtained from G by, for each $e \in E(G)$ such that $e = uv$ where $u, v \in V(G)$, deleting $E(G)$, adding a vertex w between u and v, and adding edges e_1, e_2 from u to w and e_3 between v and w. Note that this construction is not unique. Then G^* is a 2-connected 3-edge-connected bipartite plane graph with independent sets $V(G)$ and $V(G^*)\backslash V(G)$. Note that every vertex of $V(G^*)\backslash V(G)$ has only two neighbors in G^*. (See Fig. 6.2.)

Suppose that T^* is a spanning closed trail of G^*. Then for each $w \in V(G^*)\backslash V(G)$, w has either one or two neighbors on T^*. If w has two neighbors $u, v \in V(G)$ on T^*, then we let $e_w = uv \in E(G)$. Then $T = V(G) \cup \{e_w : w \in V(G^*)\backslash V(G)$ and w

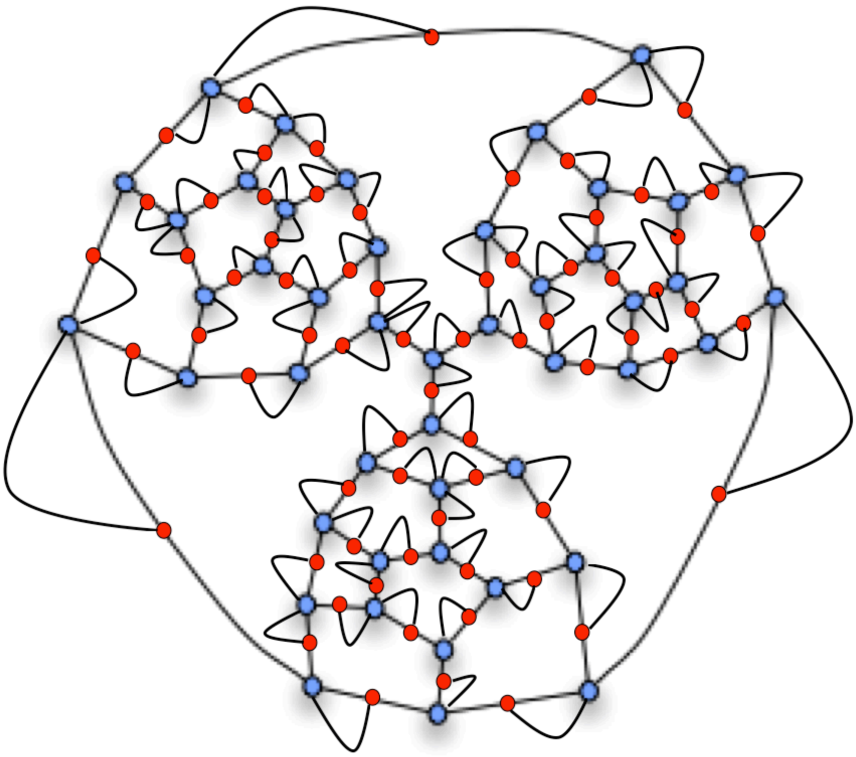


Figure 6.2: The structure of G^* when G^* is obtained from the Tutte graph G.

has two neighbors on $T^*\}$ is a Hamilton cycle of G which is a contradiction since G is non-Hamiltonian. Hence G^* has no spanning closed trail.

We give a new conjecture for 3-connected plane graphs as follows.

Conjecture 6.13 *Every 3-connected bipartite plane graph has a spanning closed trail*

This conjecture would imply Barnett's conjecture.

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